An Investigation of the Spin Groups

Gardner R. Marshall

Faculty Advisor: Dr. Randall Helmstutler

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Contents

1 Introduction

1.1 The Aims of This Paper

In this paper we discuss several fundamental concepts in algebraic topology in order to demonstrate the existence of the spin groups and investigate them topologically and algebraically. In our investigation, we demonstrate connections between these groups, the quaternions, and the Clifford algebra. We begin with the special orthogonal groups of $n \times n$ matrices and for small n produce well-known spaces with which they are homeomorphic. By carefully building an analysis of the special orthogonal groups and using some fiber bundle theory, we construct a long exact sequence from which we compute their fundamental groups. This coupled with the classification theorem for covering spaces demonstrates the existence of the spin groups. We conclude with a detailed analysis of Spin(3), showing that it is a multiplicative subgroup of the quaternions and contained within the Clifford algebra obtained from a three-dimensional vector space. With this relation we construct concrete matrix representations of the elements of Spin(3).

In addition to the purely mathematical reasons for developing and analyzing the spin groups, there is also a bit of physical motivation to do so. There is a fascinating phenomenon sometimes referred to as 4π-*periodicity* in which certain objects require not 360 but 720 degrees of rotation in order to return to their original configurations. It is this phenomenon that is used in quantum mechanics to describe why fermions are able to have half units of quantized angular momentum.

Macroscopically, the phenomenon manifests itself in objects that are connected to their environment in some special way. For these objects, one full rotation will return the object itself to its original state, but will leave its connection to the environment around it tangled. The entire system consisting of the object and its connection is definitely not the same as it was before the rotation. Further, there is no possible way, without rotating the object, to restore the system to its original state. However, suppose the object is rotated once more in the same direction as before. Despite all intuition to the contrary, instead of worsening the tangles, it is now actually possible to untangle and restore the whole system to its exact state before any rotations at all. In short, rotating the object around twice leaves it in a state essentially the same as not rotating it at all.

For instance, imagine tying one end of several strings to a chair in the center of a room. Now imagine fastening the other ends of the strings to various spots on the walls of the room. If the chair is given one full rotation, then the strings will clearly become wound up and tangled together. There is no way of fixing this without rotating the chair. If the chair is given yet another full rotation in the same direction, then the strings will appear to become even more tangled. However, it is now actually possible, without rotating the chair at all, to untangle all strings and the entire chair-string system will be exactly as it was before any rotation at all. A clever device for demonstrating this phenomenon called the "spinor spanner" is described in Ethan Bolker's article [1].

In this paper we will use what we learn about the special orthogonal groups to describe and explain this unexpected phenomenon mathematically. We will learn what it means for the system to be "essentially the same" after two full rotations in the same direction, and will see why iterated rotations can in fact undo each other. This understanding comes from the calculation of $\pi_1(SO(3))$, which we will discuss in Section 3.2.

1.2 Homotopy and the Fundamental Group

In this paper we assume the reader to be acquainted with point-set topology and introductory group theory, however there are several ideas beyond these basics that will occur throughout the paper. The reader familiar with basic algebraic topology may skip these next two sections, but for those unversed in the subject, the ideas are briefly discussed here. In this first section we introduce the concept of homotopy and define the fundamental group. In the next, we expand upon these ideas to provide an explanation of covering spaces and their role in calculating the fundamental group for various spaces. For many of the proofs and technical details, we refer the reader to [4], where things are explained in great detail.

We start with a simple idea that becomes extraordinarily useful. Let X be a space and let x_0 and x_1 be two points in X. Recall that a **path** from x_0 to x_1 in X is a continuous map $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. A **loop** in X is a path that begins and ends at the same point, that is, a path f such that $f(0) = f(1) = x_0$. The point x_0 is called the **basepoint** of the loop. If we consider two loops in a space X that have the same basepoint, then it makes sense to define a way of multiplying them by letting their product be the loop that travels through each in turn, but moves twice as quickly. In this sense we are led to consider the set of all possible loops in a space X that have a common basepoint. We denote this set by ΩX and call it the **loop space** of X. Formally, we define this loop multiplication in the following way. Given two loops f and g in X, we define the **loop product** $* : \Omega X \times \Omega X \to \Omega X$ by the equation

$$(f * g)(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is natural to ask whether or not, with this multiplication, ΩX is a group. Unfortunately it is not, however it is a large step towards the construction of a sort of "loop group." In order to step the rest of the way there, we must first introduce the concept of a homotopy and discuss what it means for two loops to be homotopic.

Definition 1.1. Let f and g be two loops in the space X with basepoint x_0. We say that f is **homotopic** to g if there is a continuous map $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = f(s)$, $H(s,1) = g(s)$, and $H(0,t) = H(1,t) = x_0$. The map H is called a **homotopy**.

At first sight this may seem to be a contrived technical construct, but upon further investigation it turns out to be quite the opposite. Two loops are homotopic if one can be continuously transformed into the other, and the homotopy is what does it. This can be seen in the definition of a homotopy: it starts at the first loop for $t = 0$, ends at the second loop for $t = 1$, and at all times t, it is itself a loop that preserves the basepoint. Note that for two loops to be homotopic they automatically must share the same basepoint.

This leads to the question of what homotopy means for two spaces. We already know that two spaces are homeomorphic if there is a continuous bijection between them whose inverse is also continuous. Another way of phrasing this is that for two homeomorphic spaces X and Y, there are continuous maps $f : X \to Y$ and $g : Y \to X$ whose compositions are equal to the identity maps on each space. We say that the two spaces are **homotopy equivalent** if this same scenario holds true except that the two composites need only be homotopic to the identity for each space rather than dead on equal to it. If two spaces X and Y are homeomorphic we will write $X \cong Y$. If two spaces are homotopy equivalent we will write $X \simeq Y$.

Example 1.1. Consider the loop in the plane $f : [0,1] \to \mathbf{R}^2$ given by $f(s) = (\cos(2\pi s), \sin(2\pi s))$. Note that the basepoint is $(1,0)$. Its image is the unit circle. Now consider the stretched and tangled loop $g : [0,1] \to \mathbf{R}^2$ given by $g(s) = (\cos(6\pi s), 3\sin(2\pi s))$. The image of this loop looks like a figure eight only with an additional section. The images of the two loops look extremely different, but the two loops themselves are in fact homotopic. This can be seen by the homotopy

$$H(s,t) = (t\cos(2\pi s) + (1-t)\cos(6\pi s), t\sin(2\pi s) + (1-t)3\sin(2\pi s)).$$

It is very easy to check that this is indeed a homotopy, and all it is doing geometrically is squashing the second more complicated loop down into the unit circle without moving the point $(1,0)$.

In thinking about homotopy it may be helpful to imagine the following analogy. Suppose you take a circular loop of potter's clay and drop it onto a flat table. It lands in the shape of a bent and lopsided circle. You then take a nail and hammer and nail down one part of the clay loop. This initial shape is like a loop in the planar table-top space (like a subset of $\mathbb{R}^2$).

Now imagine grabbing the clay loop and pulling, bending, and stretching it so that it becomes a drastically different looking loop, though it still lies flat on the table. In doing this, melding parts of the clay loop together, ripping the loop apart, moving the nailed-down part, and lifting the clay off of the table are not permitted, however it is permissible to slide some sections of the clay loop under others as they are moved around (so in this regard you are technically lifting these sections off the table if only a bit, but this is only necessary since two solid sections of clay cannot pass through each other!). After this manipulation, a much different clay loop lies on the table-top. This second loop is homotopic to the initial one since it can be formed from the first loop via the continuous manipulation that it just underwent. This manipulation even preserved the loop's basepoint (the nailed-down part). The manipulation itself is the homotopy between the two loops.

Now imagine a similar scenario, only this time the table has been freshly delivered from a hip, new modern art furniture gallery and the table-top has several large spikes pointing up from it, each many feet tall. Standing carefully above, you again drop the circular clay loop onto the table and nail down one part, but now it has fallen around a few of the spikes, capturing them inside the loop. When manipulating the clay loop in the same way as before, you are now unable to pass it through the spikes since it cannot be lifted off the table and over them. This greatly impedes your ability to alter the loop into new shapes. If a second clay loop is then dropped onto the table such that it falls around a different set of these irritating table spikes, but part of it lands over the nailed-down section of the previous loop and is itself nailed down, there will be no possible way for it to be deformed into the first clay loop unless it is lifted off the table and over the top of the spikes. These two loops are not homotopic.

So we see that the nature of the two table-tops as spaces is very different. One is very nice and simple and allows any one configuration of a clay loop to be manipulated into any other; it is what we call *simply connected*. The other table-top, the spike-riddled one, is not so simple. Given two clay loops on it, there is no guarantee that one can be manipulated into the other. This table-top has several different classes of loops that exist on its surface. The classes are each made up of the different loops that can be manipulated into each other. These classes of loops are called *homotopy classes*.

Formally, for a loop f in X, the **homotopy class** of f, denoted $[f]$, is the set of all loops in X that are homotopic to it. Now, it is not too difficult to believe and even prove that this homotopy relation is an equivalence relation, and so the homotopy classes are equivalence classes and form a partition of ΩX. For a detailed proof see §51 of [4]. Recall that the loop product fails to satisfy the group axioms for the space ΩX. It turns out however that it comes very close.

If we let f, g, and h be loops in X, for the case of associativity we have that the two loops $f * (g * h)$ and $(f * g) * h$ are not equal to each other, but are homotopic. The natural choice for an identity element is of course the constant loop e that maps all values of t to the basepoint. We find that the loops $f * e$ and $e * f$ are not equal to f, but are homotopic to it. Lastly, the inverse of a loop is the same loop travelled in reverse. For this too we have that right and left loop products of a map with its inverse are not equal to the identity loop, but are homotopic to it.

We are now about ready to make that next step which will land us at the desired "loop group." By defining a multiplication between homotopy classes by $[f] * [g] = [f * g]$, we obtain a well defined operation on the set of all homotopy classes of loops in a space. But for the homotopy classes, the relations for the group axioms become equalities and we obtain a bona fide group.

Definition 1.2. The set of all homotopy classes of loops in a space X under the loop product is called the **fundamental group** of X and is denoted $\pi_1(X)$.

Using this idea of homotopy classes is extremely useful because as we began to see in the analogy, the homotopy classes provide us a great wealth of knowledge about the nature of a space. It's not just the number of homotopy classes either that matters, but the way that they are formed from the space in question. It is the fundamental group that captures this information and organizes it mathematically. In summary, the construction goes like this: first pick a space X and a basepoint x_0 in it. Now ΩX is the set of all loops in X based at x_0. Now partition ΩX via the homotopy relation. The set of the equivalence classes is the fundamental group. It should be noted that both ΩX and $\pi_1(X)$ are in fact dependent upon choice of basepoint, but since it often makes little or no difference, the particular choice is rarely stated. For convenience, we will henceforth regard a **based space** as a space X with a preselected point $x_0 \in X$ that is the basepoint of the loops we consider in the space. We refer to x_0 as the **basepoint** of the space X as well as the loops in X, and by $\pi_1(X)$ we mean the fundamental group formed with that basepoint.

The proof that this multiplication is well defined and that it makes $\pi_1(X)$ into a group is not too difficult and can be found in §51 of [4]. The big question now is, what is the fundamental group for various spaces X? Given some topological space, we now have a method for producing a group out of it. Is this group isomorphic to any well known group? If so, how can we compute the well-known group to which it is equivalent? In this paper, when two groups G and K are isomorphic we will use the notation $G \approx K$.

Example 1.2. The fundamental group of $\mathbb{R}^n$ is trivial for all $n \geq 1$. That is, $\pi_1(\mathbb{R}^n) \approx 0$, which we use to denote the group of one element. To see this it may be helpful to recall Example 1.1. This result is an obvious generalization. Given any two loops $f, g : [0, 1] \to \mathbb{R}^n$ there exists the simple homotopy $H(s, t) = tg(s) + (1 - t)f(s)$ between them. Hence the set of all homotopy classes contains only one element and so as a group is trivial.

A path-connected space X for which $\pi_1(X)$ is trivial is said to be **simply connected** (recall the flat table-top). Hence we have encountered our first family of simply connected spaces. Supposing we have a continuous map between two topological spaces, how can we preserve the information that this map provides? The answer is fairly simple, and also very nice. Before we elaborate however, let us decide now that henceforth all maps in consideration are to be basepoint-preserving unless stated otherwise. This will be of great notational convenience.

Definition 1.3. Let X and Y be based spaces and let $f : X \to Y$ be a continuous map. The map $f_* : \pi_1(X) \to \pi_1(Y)$ defined by $f_*([\alpha]) = [f \circ \alpha]$ is called the **induced homomorphism**.

This map is indeed a homomorphism, but it also has two important properties. First, the starring process preserves identity maps. That is, if 1 is the identity map on X, then the map 1_* is the identity map on $\pi_1(X)$. Second, starring preserves composition, i.e. for any two composable maps f and g we have $(f \circ g)_* = f_* \circ g_*$. From this we see that if $X \cong Y$ then $\pi_1(X) \approx \pi_1(Y)$. Once again, the proofs can be found in §52 of [4].

Since loops begin and end at the same point, one can think of them as mapping out of the unit circle S^1 instead of the unit interval. This is because the two endpoints are "glued" together in the sense that they both map to the basepoint. In this light $\pi_1(X)$ is the group of homotopy classes of loops $S^1 \to X$. This suggests a natural extension to higher spheres. We define $\pi_n(X)$ to be the set of homotopy classes of loops $S^n \to X$. It too is a group for the same reasons.

Unfortunately, computing the fundamental group is extremely difficult for all but the simplest spaces. There are many tools and techniques for attacking this problem, and in the next section we introduce one of the most fundamental. It is the idea of a covering space.

1.3 Covering Spaces

As its name suggests, given a space B, a covering space may be thought of loosely as a space E that floats above and "covers" B overhead. The points in E essentially project down to points in B, though the way in which this must happen is special. Because of the special nature of a covering space, it is sometimes easier to compute its fundamental group than that of the space it covers. This can be a useful tool in calculating the fundamental group of the original space.

Definition 1.4. Let $p : E \to B$ be a continuous surjection between based spaces. Suppose that for every point $b \in B$ there is a neighborhood U_b of b such that $p^{-1}(U_b) = \coprod_i V_i$, where for each open set V_i, the restriction map $p|_{V_i} : V_i \to U_b$ is a homeomorphism. We say that p is a **covering map** and that E is a **covering space**, or simply a **cover** for short. The space B is referred to as the **base**, and if b_0 is the basepoint of B, then we call its inverse image $p^{-1}(b_0)$ the **fiber** of p. Lastly, if the space B is connected, the order of the fiber is said to be the **number of sheets** of the cover.

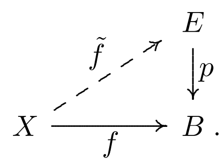
The essential feature of a covering space is that locally each spot looks just like some spot of its base. This is accounted for by the requirement that the restriction to each of the neighborhoods V_i be a homeomorphism. Globally however, the cover may be a drastically different kind of shape altogether. So a cover captures the features of its base, but can do this without being a replica and is thus allowed to be much bigger and more "filled in." This is why it can actually be a simpler space from the view of the fundamental group.

Example 1.3. The space $X \times \{1, 2, 3\}$ is a cover of the space X and the covering map is the ordinary projection map $p(x, n) = x$. This is an example of a *trivial cover* because the covering space is merely a stack of disjoint copies of the base space. In order to prevent such uninteresting covers, we usually desire our covering spaces to be connected.

Example 1.4. The quintessential example of a nontrivial cover is the real line over the circle. Imagine coiling the real line around and around to produce an infinitely long slinky. It is easy to see that projecting the points on this infinite coil down results in a circle. Additionally, for each point on this circle there is a little arc about it for which there are infinite distinct and homeomorphic copies on the coiled real number line. The exact map $p : \mathbb{R} \to S^1$ is given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$.

One of the most important ideas related to covering spaces is that of a lift. It will be used time and time again.

Definition 1.5. Let $p : E \to B$ be a covering map and let $f : X \to B$ be a continuous map. A continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ is called a **lift**. In short, $\tilde{f}$ is a continuous map along the dashed arrow that makes the following diagram commute:



One of the most useful aspects of lifts is that they are often unique once it has been decided where they send the basepoint of the space out of which they map. The following proposition tells us exactly when this is true.

Proposition 1.1. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$ and let $f : X \to B$ with $f(x_0) = b_0$ be a continuous map. Let X be path-connected and locally path-connected. Then the map f has a unique lift $\tilde{f} : X \to E$ satisfying $\tilde{f}(x_0) = e_0$ if and only if $im(f_*) \subseteq im(p_*)$.*

For proof, see Lemma 79.1 of [4]. In particular, if the space X is simply connected, then $\operatorname{im}(f_*) = 0$ and hence is a subset of $\operatorname{im}(p_*)$ regardless of p. We will repeatedly make use of this throughout the paper so we note it here as a corollary.

Corollary 1.2. *Suppose $p : E \to B$ is a covering map with $p(e_0) = b_0$ and $f : X \to B$ is a continuous map with $f(x_0) = b_0$. If X is simply connected and locally path-connected, then f has a unique lift $\tilde{f} : X \to E$ satisfying $\tilde{f}(x_0) = e_0$.*

We now introduce one final tool for computing the fundamental group. It will be used later and plays an important role in synthesizing the ideas introduced thus far.

Definition 1.6. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$, and let $\tilde{f}$ be the lift of a loop f in B such that $\tilde{f}(0) = e_0$. The **lifting correspondence** is the map $\phi : \pi_1(B) \to p^{-1}(b_0)$ defined by $\phi([f]) = \tilde{f}(1)$.

The lifting correspondence is well defined since given two representatives of a homotopy class in $\pi_1(B)$, their lifts are homotopic and thus end at the same point.

Proposition 1.3. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. If E is simply connected, then the lifting correspondence $\phi : \pi_1(B) \to p^{-1}(b_0)$ is a bijection (of sets only).*

The proof is fairly simple, and the result can be used in conjunction with Example 1.4 to show that $\pi_1(S^1) \approx \mathbb{Z}$. To see this the reader is referred to §54 of [4]. In fact, for $k < n$ we have $\pi_k(S^n) \approx 0$. This is not too hard to see. Consider the case for $\pi_1(S^2)$. Any loop on the surface of the two-sphere can be shrunk down on the surface of the sphere until it becomes a point. The loops are based and so it is fairly clear that any two loops can be continuously deformed into one another and are homotopic. The case is similar for higher orders.

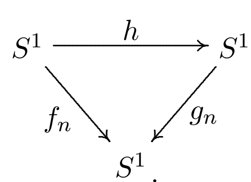
Suppose $p : E \to B$ is a cover with $\pi_1(E) \approx 0$. We call E the **universal cover** of B. It is considered the largest and most encompassing cover of a space. There is now one final but massive hurdle before the close of this section. It is the classification of covering spaces, which puts classes of covering spaces of a base space in one-to-one correspondence with subgroups of the fundamental group of the given base. Its proof is arduous, and the result is anything but trivial. It requires a bit of extra information and additional definitions for which there is not sufficient time to explain here in detail. The main idea is all that we need. It can be shown that given a covering map $p : E \to B$, its induced homomorphism $p_* : \pi_1(E) \to \pi_1(B)$ is injective. This means that $\text{im}(p_*)$ is a subgroup of $\pi_1(B)$ isomorphic to $\pi_1(E)$. We denote this subgroup by H_p.

Theorem 1.4 (Classification Theorem)**.** *Let B be a path-connected, locally path-connected, semi-locally simply connected space. Let $K(B)$ be the set of unbased equivalence classes of covers of B and denote such a class by $\langle p \rangle$. Let $H(B)$ be the set of conjugacy classes of subgroups of $\pi_1(B)$ and denote such a class by $[H]$. Then the classification map $\xi : K(B) \to H(B)$ defined by $\xi(\langle p \rangle) = [H_p]$ is a bijection.*

The proof is not given here, but is the culmination of the entire chapter 13 of Munkres [4], to which we refer the reader who is interested in learning the details and checking the proof.

Example 1.5. Consider the circle S^1. We have that $\pi_1(S^1) \approx \mathbb{Z}$, hence it has an infinite number of subgroups. Specifically, it has a subgroup $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$, and these are all of the subgroups of $\mathbb{Z}$. Each of these is the only subgroup in its conjugacy class. According to the Classification Theorem, there is a cover for each of these subgroups. In fact, each cover is the complex exponential map $f_n : S^1 \to S^1$ given by $f_n(z) = z^n$, where we treat S^1 as a subspace of $\mathbb{C}$ and use complex multiplication. These maps are covering maps of n sheets because they wrap the circle around itself n times. The map that corresponds to the trivial subgroup, i.e. $n\mathbb{Z}$ for $n = 0$, is the map given in Example 1.4. In other words, the covering space corresponding to the smallest possible subgroup 0 is the largest possible cover $\mathbb{R}$.

But what of the maps $g_n : S^1 \to S^1$ given by $g_n(z) = z^{-n}$? These too are covering maps, which wrap the circle around itself n times. They merely wrap in the reverse orientations. By the Classification Theorem, the unbased equivalence classes of covers ought to be in one-to-one correspondence with the conjugacy classes of subgroups of $\mathbb{Z}$. As it turns out, each map g_n is in fact equivalent to the map f_n. We have this *unbased equivalence* of covers because for each pair of maps f_n and g_n, there is a homeomorphism h that makes the following diagram commute:



The homeomorphism is simply the restriction of the conjugation map $h : \mathbb{C} \to \mathbb{C}$ to the circle S^1. That is, $h(z) = \bar{z}$. In other words, for each $n \in \mathbb{N}$ we have $-n\mathbb{Z} = n\mathbb{Z}$ and $g_n \simeq f_n$. Hence we have classified all of the covers of S^1 by means of the Classification Theorem.

With this background material laid out, we are now ready to begin an analysis of the special orthogonal groups. We will see that their fundamental groups are isomorphic to $\mathbb{Z}_2$ for $n > 2$.

Hence by the Classification Theorem, they will each have two covers, one universal and one trivial. The spin groups are the universal covers.

2 The Special Orthogonal Groups

2.1 The Definition and Basic Properties

For each positive integer n, the special orthogonal group $SO(n)$ is the group of rotations in $\mathbb{R}^n$. These groups arise in both pure and applied mathematics and are extremely useful in the study of physics. In this section we will define them formally and discuss their properties. Later, we will use these properties to compute their fundamental groups, which coupled with the Classification Theorem 1.4 guarantees the existence of the spin groups. Before we begin, it is first helpful to introduce the concept of a topological group.

Definition 2.1. Suppose G is both a group and a topological space. If both the group multiplication $G \times G \to G$ and the inversion map $G \to G$ are continuous, then G is said to be a **topological group**.

Topological groups can be loosely thought of as being locally homogeneous. That is, upon selecting any two spots in the space, they can be made to appear the same by "zooming in" on both. This homogeneity is a result of the imposed group structure, whose nature is necessarily continuous. With this in mind, we turn to the special orthogonal groups.

Definition 2.2. The set of all $n \times n$ matrices A for which $\det(A) = 1$ and $A^T A = AA^T = I$ is called the **special orthogonal group** and is denoted $SO(n)$.

The elements of special orthogonal groups have many useful attributes which can be easily derived directly from the definition. For instance, note that for $A \in SO(n)$ its inverse A^{-1} is the same as its transpose A^T. As we shall see, in accordance with their name, the special orthogonal groups are in fact groups, but they are also more. They behave nicely as topological objects as well.

Lemma 2.1. *The set GL_n of $n \times n$ matrices with nonzero determinant is a topological group under matrix multiplication.*

Proof. It is a simple matter to show that GL_n is a group. It is closed under the operation because the determinant is multiplicative, and every matrix in it has an inverse since its determinant is nonzero. Matrix multiplication is associative, and GL_n clearly contains the identity matrix, hence it is a group. We can now easily topologize GL_n as a subspace of $\mathbb{R}^{n^2}$ by thinking of its entries as coordinates. Further, matrix multiplication is inherited from M_n, where it consists of taking polynomials in the entries of the two matrices being multiplied as the entries of the resultant matrix, and thus is continuous. Similarly, inverting matrices consists of the inverse of the determinant map, which for matrices in GL_n is never zero, and the matrix of cofactors, which are again obtained via polynomials in the entries. Hence inversion is also continuous, and so GL_n is a topological group. □

Proposition 2.2. *The set $SO(n)$ is a topological group under matrix multiplication.*

Proof. The set $SO(n)$ is clearly a subset of GL_n that contains the identity and is closed under inverses. It is equally simplistic to see that it is closed under the operation since again, the determinant is multiplicative. Since it can be topologized as a subspace of GL_n and inherits the same continuous multiplication and inversion, it is a topological group. □

Taking the following proposition as a starting point, we will now begin to branch out into the many aspects of $SO(n)$. The next proposition has the added bonus of illustrating the appropriateness of the special orthogonal group's name.

Proposition 2.3. *The columns of a fixed matrix in $SO(n)$ constitute a set of orthogonal unit vectors in $\mathbb{R}^n$.*

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ represent two matrices in $SO(n)$. We have that $AB = (c_{ij})$, where $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$. Since $A^T A = I$ for all matrices in $SO(n)$, we have

$$A^T A = \left(\sum_{k=1}^n a_{ki}a_{kj}\right) = (\delta_{ij}).$$

This means two things. First, if $i = j$, then $\sum_{k=1}^n a_{ki}^2 = 1$. In other words, the sum of the squares of the terms in the i^{th} column is one, and so the columns are unit vectors. Secondly, if $i \neq j$, then $\sum_{k=1}^n a_{ki}a_{kj} = 0$. This indicates that the dot product of two columns is zero, i.e. the columns are orthogonal to each other and make up an orthonormal basis for $\mathbb{R}^n$.

□

From here we may begin to see why this group of matrices is the group of length preserving rotations in $\mathbb{R}^n$. A matrix $A \in SO(n)$ represents a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Let $\vec{v}$ be a vector in $\mathbb{R}^n$, and let $\vec{e_i}$ be the i^{th} unit vector in the standard basis for $\mathbb{R}^n$. In other words, $\vec{e_i}$ is the vector of all zeros, except at the i^{th} spot, where there is a one. We can write $\vec{v}$ as a linear combination of these unit vectors, $\vec{v} = \sum_{i=1}^n c_i\vec{e_i}$. Noting that A represents a linear transformation and that $A\vec{e_i} = \vec{a_i}$ is the i^{th} column of A, we have

$$A\vec{v} = A\sum_{i=1}^n c_i\vec{e_i} = \sum_{i=1}^n c_iA\vec{e_i} = \sum_{i=1}^n c_i\vec{a_i}.$$

But since the columns of A are orthogonal unit vectors, the set $\{\vec{a_i}\}_{i=1}^n$ constitutes an orthonormal basis for $\mathbb{R}^n$. Thus we have

$$|\vec{v}| = \left(\sum_{i=1}^n c_i^2\right)^{\frac{1}{2}} = |A\vec{v}|$$

and so A preserves the length. Hence A must be either a rotation or a reflection. But since $\det(A) = 1$, it also preserves orientation, and so A must represent a length preserving rotation in $\mathbb{R}^n$.

At this point we will simply run through a list of propositions and details about $SO(n)$ which will come in handy later, and which ultimately lead to a method for calculating the fundamental group of $SO(n)$.

Proposition 2.4. *Define a map $g : SO(n-1) \to SO(n)$ by $g(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then g is a continuous, injective homomorphism, and thus $SO(n-1)$ may be embedded into $SO(n)$ as a subgroup.*

Proof. We first verify that g maps into $SO(n)$. We have that $\det(g(A)) = \det(A) = 1$, but also

$$g(A)^T g(A) = \begin{pmatrix} A^T & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^T A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix},$$

hence $g(A) \in SO(n)$. Now it is clear that g is injective, and since g is simply an embedding of a subspace into $SO(n)$, it is continuous as well. Lastly, we have that

$$g(AB) = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = g(A)g(B),$$

and so g is a homomorphism.

□

Let $SO(n)/SO(n-1)$ be the set of all left cosets of $SO(n-1) \subseteq SO(n)$. Topologize it in the quotient topology under the quotient map $q : SO(n) \to SO(n)/SO(n-1)$ that sends each element $A \in SO(n)$ to its left coset $ASO(n-1)$. Note that this quotient space is a topological space, but not necessarily a group.

Proposition 2.5. *Both $SO(n)$ and $SO(n)/SO(n-1)$ are compact spaces.*

Proof. Recall that $SO(n)$ is a subspace of $\mathbb{R}^{n^2}$. For every $A \in SO(n)$ we have that each of its column vectors is a unit vector, i.e. each column vector lies in S^{n-1}, hence $SO(n)$ is a bounded subspace of $\mathbb{R}^{n^2}$. Additionally, $\det(A) = 1$ and so $SO(n)$ is a closed and bounded subspace of a Euclidean space and hence compact. Now the quotient map $q : SO(n) \to SO(n)/SO(n-1)$ is continuous and surjective, and so $SO(n)/SO(n-1)$ is compact as well.

□

Proposition 2.6. *For $n \geq 2$ define a map $f : SO(n) \to S^{n-1}$ by $f(A) = A\mathbf{e}_n$ where $\mathbf{e}_n = (0, 0, \ldots, 1)^T$. That is, f sends a matrix $A \in SO(n)$ to its last column, which is a unit vector and hence in S^{n-1}. Then f is a continuous surjection.*

Proof. Let $\vec{v} \in S^{n-1}$. We want $A \in SO(n)$ such that $\vec{v}$ is its last column. That is, we want a set of $n-1$ orthonormal vectors in $\mathbb{R}^n$ all of whom are orthogonal to $\vec{v}$. Now there exists a basis for $\mathbb{R}^n$ with $\vec{v}$ in it. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{v}\}$ be such a basis. Consider the matrix $C = (\ \vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_{n-1} \mid \vec{v}\ )$. If $\det(C) = -1$ swap $\vec{v}_1$ and $\vec{v}_2$ so that $\det(C) = 1$, otherwise do nothing. Now apply the Gram-Schmidt procedure in reverse beginning with $\vec{v}$ to obtain an orthonormal basis out of $\mathcal{B}$. Since $\vec{v} \in S^{n-1}$, it is already a unit vector and is unchanged as the first vector in the procedure. This process of Gram-Schmidt in reverse does not change the determinant of the matrix C since it applies only the elementary column operation involving adding scalar multiples of columns at each step. Now we take as our desired A the matrix obtained by taking the vectors in this orthonormal basis as columns with $\vec{v}$ last. It is an element of $SO(n)$ and we have $f(A) = \vec{v}$, hence f is surjective. It is clear that f is continuous since f is defined by matrix multiplication and is merely a projection.

□

For the next proposition it will be useful to prove the following lemma.

Lemma 2.7. *Let $f : SO(n) \to S^{n-1}$ be the map from Proposition 2.6. If $A \in SO(n)$ and $B \in SO(n-1)$, then $f(AB) = f(A)$.*

Proof. Suppose $A \in SO(n)$ and $B \in SO(n-1)$ and denote A and B by (a_{ij}) and (b_{ij}) respectively. We have $AB = (c_{ij})$ with $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, hence

$$f(AB) = \left(\sum_{k=1}^{n} a_{1k}b_{kn} \ , \ \sum_{k=1}^{n} a_{2k}b_{kn} \ , \ \ldots \ , \ \sum_{k=1}^{n} a_{nk}b_{kn} \right)^T .$$

However, since $B \in SO(n-1)$ we have

$$\begin{aligned} b_{in} &= 0, \quad 1 \le i < n \\ b_{nn} &= 1, \\ b_{nj} &= 0, \quad 1 \le j < n \end{aligned}$$

and so $f(AB) = (a_{1n}, a_{2n}, \ldots, a_{nn})^T = f(A)$.

□

Proposition 2.8. *If $\vec{v} \in S^{n-1}$ then $f^{-1}(\vec{v})$ is a left coset of $SO(n-1)$.*

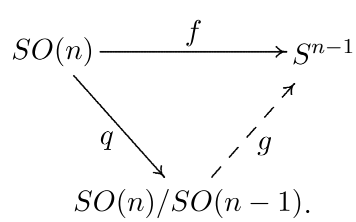
Proof. Let $\vec{v} \in S^{n-1}$, and choose $A \in SO(n)$ such that $f(A) = \vec{v}$. We will show that $f^{-1}(\vec{v}) = ASO(n-1)$. Suppose $AB \in ASO(n-1)$ for $B \in SO(n-1)$. By Lemma 2.7 we have $f(AB) = f(A) = \vec{v}$, hence $ASO(n-1) \subseteq f^{-1}(\vec{v})$. Now, suppose $C \in f^{-1}(\vec{v})$. We want $C = AB$ for some $B \in SO(n-1)$. Let $B = A^{-1}C = A^TC$. Since $A^TC \in SO(n)$, every column is a unit vector and so it suffices to show that the (n,n) entry of A^TC is 1 in order to show that $B = A^TC \in SO(n-1)$.

Now, since $f(A) = \vec{v}$ we have that the n^{th} row of A^T is $\vec{v}$, but also $C \in f^{-1}(\vec{v})$ so the n^{th} column of C is $\vec{v}$ as well. But the (n,n) entry of A^TC is given by the dot product of the n^{th} row of A^T and the n^{th} column of C. Hence this entry is given by $\vec{v} \cdot \vec{v} = 1$, as desired. Thus $B \in SO(n-1)$ and we have $ASO(n-1) = f^{-1}(\vec{v})$.

□

Proposition 2.9. *The map $f : SO(n) \to S^{n-1}$ induces a homeomorphism $SO(n)/SO(n-1) \cong S^{n-1}$.*

Proof. Let $q : SO(n) \to SO(n)/SO(n-1)$ be the quotient map, sending each matrix A in $SO(n)$ to the left coset containing it. For all $B \in q^{-1}(ASO(n-1))$ we know that $B \in ASO(n-1)$. That is, $B = AC$ for some $C \in SO(n-1)$. Hence by Lemma 2.7 we have that $f(B) = f(AC) = f(A)$ and so f is constant on $q^{-1}(ASO(n-1))$. Thus, by the universal property of the quotient map, we know that the map g in the following diagram exists and is continuous:



We claim that g is a homeomorphism. Because f is surjective and the diagram commutes, g is surjective as well. Now if $g(ASO(n-1)) = g(BSO(n-1))$ then $g(q(A)) = g(q(B))$, which implies that $f(A) = f(B)$, i.e. the last columns of A and B are the same. Now, by Proposition 2.8 we have that $BSO(n-1) = f^{-1}(\vec{v})$ where $\vec{v}$ is the last column of B, but A is certainly in $f^{-1}(\vec{v})$ and so $A \in BSO(n-1)$. This means that $ASO(n-1) = BSO(n-1)$, and so g is injective. Finally, from Proposition 2.5 we have that $SO(n)/SO(n-1)$ is compact, and S^{n-1} is certainly Hausdorff. There is a theorem from point-set topology which states that a continuous bijection that maps a compact space into a Hausdorff space is a homeomorphism; for the interested reader it is Theorem 26.6 of [4]. Hence g is a homeomorphism.

□

We will see that the homeomorphism $SO(n)/SO(n-1) \cong S^{n-1}$, along with the map f, turn out to be extremely useful. This is because they will be used to construct a fiber bundle giving $\pi_1(SO(n))$, which we will discuss in detail later. For now however, we first turn to the question of how $SO(n)$ behaves as a space for small n values.

2.2 Low-Dimensional Homeomorphisms

For the case where $n = 1$, the group $SO(n)$ is not very interesting; it is merely a point. But what about higher values? How does it change? For the plane $\mathbb{R}^2$, the group of rotations can be thought of as the circle. Each angle corresponds to a possible rotation on the plane, hence $SO(2) \cong S^1$ as spaces. But they are also isomorphic as groups, where the group structure of S^1 is that which it attains as a multiplicative subgroup of the complex plane. The case for $\mathbb{R}^3$ is a bit more complicated. The following is a brief paraphrasing of the explanation Allen Hatcher provides in section 3.D of his book [2].

Let B_3 be the solid unit ball in $\mathbb{R}^3$. Consider the map $\theta : B_3 \to SO(3)$ defined by mapping a nonzero point $\vec{v} \in B_3$ to the rotation in $\mathbb{R}^3$ of angle $|\vec{v}|\pi$ about the axis formed by the line passing through $\vec{v}$ and the origin. The zero point is sent to the identity rotation. In doing this, we select the convention of the right-hand rule in order to eliminate any ambiguity.

Now $\mathbb{R}P^3$ can be viewed as the quotient space of B_3 obtained by identifying antipodal points on the boundary to the same point. Let $q : B_3 \to \mathbb{R}P^3$ be the quotient map. Now antipodal points on the boundary of B_3 are sent to the same rotation by the map θ, and so by the universal property of the quotient map, θ induces a continuous map $\vartheta : \mathbb{R}P^3 \to SO(3)$ as illustrated in the following diagram:

$$\begin{array}{ccc} B_3 & \xrightarrow{\theta} & SO(3) \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle \vartheta} & \\ & \mathbb{R}P^3. & \end{array}$$

The map ϑ is injective because the axis of every rotation is given uniquely by the origin and $\vec{v}$. It is surjective because every nonzero rotation must occur about some axis. Hence by once again using Theorem 26.6 of [4] we have that ϑ is a homeomorphism and $SO(3) \cong \mathbb{R}P^3$. We will state this as a proposition.

Proposition 2.10. *The space $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.*

From this we are able to obtain the following proposition.

Proposition 2.11. *The fundamental group of $SO(3)$ is isomorphic to $\mathbb{Z}_2$.*

Proof. Consider the quotient map $q : S^3 \to \mathbb{R}P^3 \cong SO(3)$ that sends antipodal points on S^3 to the same point. It is a covering map of two sheets. To see this, consider a point in $\mathbb{R}P^3$. There is a neighborhood around that point whose inverse image under the quotient map is two distinct copies of it. One must simply be careful not to select a neighborhood too large, i.e. one for which the inverse image contains a pair of antipodes.

Now, S^3 is a simply connected space because it is path-connected and $\pi_1(S^3) \approx 0$. Hence by Proposition 1.3, the lifting correspondence $\phi : \pi_1(SO(3)) \to q^{-1}(b_0)$ is a bijection. But $|q^{-1}(b_0)| =$

2, and because there is only one group with two elements up to isomorphism, we have $\pi_1(SO(3)) \approx \mathbb{Z}_2$.

$\square$

We have now calculated our first fundamental group of a special orthogonal group, and as we will see in Section 3.2, this explains 4π-periodicity. In order to calculate the rest, we will need to know this for $SO(3)$ as a base case.

3 The Existence of the Spin Groups

3.1 A Tiny Bit of Bundles

In order that we may calculate the fundamental group of $SO(n)$, we will need to employ tools from the theory of fiber bundles. Fiber bundles are generalizations of covering spaces to permit for continuous fibers. Imagine the same idea of a covering space floating above its base, but now instead of several distinct sheets overhead, imagine a continuous chunk of "sheets."

Definition 3.1. Let $p : E \to B$ be a continuous surjection between based spaces, and let $F = p^{-1}(b_0)$. Suppose for every point $b \in B$ there is a neighborhood U of b and a homeomorphism $h_U : p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h_U} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U & \end{array},$$

where p_1 is the projection map on the first factor. Then we say that p is a **fiber bundle** with **fiber** F, and we call $F \xrightarrow{i} E \xrightarrow{p} B$ a **fiber sequence**, where $i : F \to E$ is the inclusion map.

Example 3.1. One example of a fiber bundle that we already know is a covering map. A cover is a fiber bundle with a discrete fiber. Analogous to a trivial cover is a trivial fiber bundle, which instead of being a stack of discrete copies of the base, is a continuous "strip" or chunk of copies of the base. In other words, the projection map on the first factor $p_1 : B \times F \to B$ is a trivial fiber bundle. For instance, the projection map from the cylinder down to the circle is a trivial bundle.

Example 3.2. The map that projects points on the Möbius strip down to its meridian circle is a fiber bundle over the circle S^1. The fiber is an interval across the width of the Möbius strip, dependent of course upon choice of basepoint.

Example 3.3. The map $f : SO(n) \to S^{n-1}$ that maps $A \in SO(n)$ to its last column, which we saw in Proposition 2.6, is a fiber bundle. This will turn out to be of great use to us later.

Before we continue with bundles, we will need to introduce the notion of an exact sequence, which is essentially a long string of groups with homomorphisms mapping between them. The special feature of an exact sequence is that at each step, i.e. at each group, the image of the incoming map is equal to the kernel of the outgoing map. The following is a formal definition of an exact sequence of groups.

Definition 3.2. An **exact sequence** of groups is a sequence of groups and homomorphisms between them

$$G_1 \xrightarrow{\theta_1} G_2 \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_{n-1}} G_n,$$

such that $\mathrm{im}(\theta_i) = \ker(\theta_{i+1})$ for $1 \leq i < n - 1$.

Exact sequences are useful because their setup provides much information about the groups out of which they are made. Once constructed from known groups, we may infer information about their parts by how they fit together. The following is an example of how exact sequences can be helpful. We will in fact use this later as well.

Remark. Suppose we have the following two exact sequences of groups and homomorphisms

$$0 \xrightarrow{\alpha} A \xrightarrow{\beta} B \quad \text{and} \quad C \xrightarrow{\gamma} D \xrightarrow{\delta} 0\,.$$

We first know that $\alpha = 0$ since it is a homomorphism and must map the identity to the identity. Secondly, we know that $\delta = 0$ since its image is trivial. From this we know that β is injective since $\mathrm{im}(\alpha) = 0 = \ker(\beta)$. We also know that γ is surjective because $\mathrm{im}(\gamma) = \ker(\delta) = D$. If we combine these conditions by sandwiching two groups between 0's then we find that the homomorphism between them is bijective and hence an isomorphism.

Exact sequences can be very informative, but one may wonder how to go about constructing them. After all, they aren't as helpful if they require knowing all of the information beforehand. In fact there are clever ways of indirectly forcing exact sequences into existence. The following theorem is an excellent example and will be a crucial step for calculating $\pi_1(SO(n))$.

Theorem 3.1. *If* $F \xrightarrow{i} E \xrightarrow{p} B$ *is a fiber sequence, then there exists a homomorphism* $\partial : \pi_n(B) \to \pi_{n-1}(F)$ *such that the following sequence is exact:*

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{i_*} \pi_{n-1}(E) \longrightarrow \cdots\,.$$

The proof may be found in Appendix A.

3.2 The Fundamental Group of $SO(n)$

We are now fully equipped to tackle the long awaited problem of finding $\pi_1(SO(n))$ for $n \geq 3$.

Theorem 3.2. *The fundamental group of* $SO(n)$ *is isomorphic to* $\mathbb{Z}_2$ *for* $n \geq 3$.

Proof. Taking $\pi_1(SO(3)) \approx \mathbb{Z}_2$ as our base case, which we have by Proposition 2.11, we proceed by induction on n. Suppose $\pi_1(SO(n-1)) \approx \mathbb{Z}_2$. We wish to show that $\pi_1(SO(n)) \approx \mathbb{Z}_2$. Let $f : SO(n) \to S^{n-1}$ be the map from Proposition 2.6. For a closed subgroup H inside a topological group G, the quotient map $q : G \to G/H$ is a fiber bundle with fiber H. But by Proposition 2.9 we have that $SO(n)/SO(n-1) \cong S^{n-1}$ and so the sequence

$$SO(n-1) \xrightarrow{i} SO(n) \xrightarrow{f} S^{n-1}$$

is a fiber sequence as $SO(n-1)$ is a closed subgroup of $SO(n)$. Hence by Theorem 3.1 we have the following segment of the long exact sequence:

$$\pi_2(S^{n-1}) \xrightarrow{\partial} \pi_1(SO(n-1)) \xrightarrow{i_*} \pi_1(SO(n)) \xrightarrow{f_*} \pi_1(S^{n-1})\,.$$

But since $\pi_1(SO(n-1)) \approx \mathbb{Z}_2$ and $\pi_2(S^{n-1}) \approx 0 \approx \pi_1(S^{n-1})$ for $n \geq 4$ we have

$$0 \xrightarrow{\partial} \mathbb{Z}_2 \xrightarrow{i_*} \pi_1(SO(n)) \xrightarrow{f_*} 0\,.$$

Since the two groups are now sandwiched between 0's in an exact sequence, we have that $\pi_1(SO(n)) \approx \mathbb{Z}_2$ as desired (recall the remark above). This completes the proof.

□

We are now in an excellent position to begin to understand the 4π-periodicity phenomenon in the context of homotopy theory. Consider the example of the chair given in the introduction. It has many strings attaching it to the walls of the room. The room of course exists in a space much like $\mathbb{R}^3$, and so rotations of the chair-string system are like elements of $SO(3)$.

We have learned that the fundamental group of $SO(3)$ is isomorphic to $\mathbb{Z}_2$. In the case of our chair-string example, leaving the system untangled (no rotation of the system) corresponds to the zero element of $\mathbb{Z}_2$. It can be thought of as the constant loop at the basepoint; it simply remains at the zero rotation. On the other hand, one full rotation in some direction corresponds to the nonzero element of $\mathbb{Z}_2$. It can be thought of as a loop that travels through every rotation about some fixed axis until it arrives back at the zero rotation again. Multiplying this loop with itself via the loop product simply consists of traversing it twice. But we have that $[1] + [1] = [0]$ inside $\mathbb{Z}_2$ and hence we expect that rotating the chair around twice in the same direction leaves the chair-string system essentially the same as it began.

What does it mean for the chair-string system to be "essentially the same" as it began? It means that the loop resulting in its configuration is homotopic to the constant loop at the zero rotation, where it began. In terms of the physical nature of the chair-string system, this means that it is possible to untangle the strings without rotating the chair at all. It is this untangling that is the homotopy.

In short, we have completely classified all of the possible "tangle classes" of the chair-string system. There are only two, and they are the elements of $\mathbb{Z}_2$. That is, regardless of how the chair is rotated, the resulting tangle is always able to be untangled into either the configuration resulting from a single rotation, or the original configuration resulting from no rotation (no tangles). Further, repeated rotations in the same direction result in moving back and forth between these "tangle classes," which we secretly know to be the homotopy classes of $\pi_1(SO(3))$.

As an interesting sidelight, we now know that the same result holds for all higher spatial dimensions as well. This is because $\pi_1(SO(n)) \approx \mathbb{Z}_2$ for $n \geq 3$. However, for the case $n = 2$ we have that $\pi_1(SO(2)) \approx \mathbb{Z}$. This means that if we restrict ourselves to the plane, we can never untangle the strings and return the system to its original configuration without rotating the chair back. No matter how many times we rotate the chair in the same direction, we will simply continue to enter new and different "tangle classes." This is because $\mathbb{Z}$ is infinite cyclic and regardless of how many times we add $1 \in \mathbb{Z}$ we will never obtain 0 and we will always obtain a different element.

In this way $\mathbb{R}^2$ is "too small" to permit 4π-periodicity. Most of us would imagine that $\mathbb{R}^3$ works the same way $\mathbb{R}^2$ does, and that with every rotation the tangle only becomes worse. However, as we have seen this is not the case. There is in fact sufficient freedom within the space we live to untangle the strings after two complete rotations. For ideas on how to construct a device to try this out, and for hints on how to untangle the strings, we refer the reader to [3] and [1].

3.3 The Spin Groups

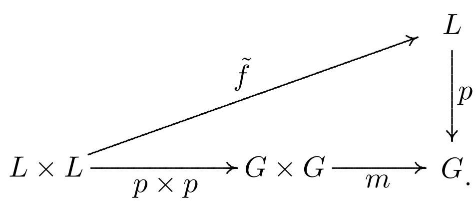
We are now, at long last, ready to define the spin groups. For the group $SO(n)$, we are guaranteed the existence of a covering space for every subgroup of $\pi_1(SO(n))$ by the Classification Theorem 1.4. But we know $\pi_1(SO(n)) \approx \mathbb{Z}_2$ for $n \geq 3$ and so there are two subgroups of the fundamental group, which are the trivial subgroup 0 and the entire group $\mathbb{Z}_2$. The cover corresponding to the entire group is the trivial cover, i.e. a single copy of $SO(n)$. The cover that corresponds to the trivial subgroup is the universal cover of $SO(n)$. It is a double cover because $\mathbb{Z}_2$ is a group of two elements. This universal double cover is the spin group $\mathrm{Spin}(n)$.

Definition 3.3. For $n \geq 3$, the unique universal double cover of the topological group $SO(n)$ is called the **spin group** and is denoted $\mathrm{Spin}(n)$.

At this point, the spin groups are merely spaces, but we will show shortly that they are in fact groups. The spin groups are not defined for $n = 1, 2$. For the case $n = 1$ we have that $SO(1) \cong 0$ and $\pi_1(SO(1)) \approx 0$, hence there is no universal double cover of $SO(1)$, only trivial covers. In short $SO(1)$ is about as boring as possible. For the case $n = 2$ recall that we have $SO(2) \cong S^1$. But $\pi_1(S^1) \approx \mathbb{Z}$ and so it too has no universal double cover since its universal cover is $\mathbb{R}$, which has a fiber of infinite order. For the rest of the special orthogonal groups there is a corresponding spin group, and as it turns out, the spin groups are topological groups.

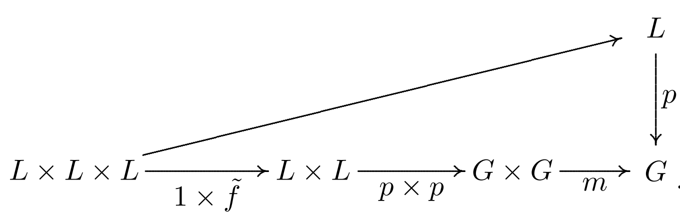
Theorem 3.3. *Let G be a locally path-connected topological group and let $p : L \to G$ be its universal cover. Given a point in the fiber, L admits a unique topological group structure for which p is a homomorphism.*

Proof. Let $m : G \times G \to G$ be the group multiplication in G. Choose the identity $e \in G$ as the basepoint of G and choose $e' \in p^{-1}(e)$ to be the basepoint of L. Consider the map $f : L \times L \to G$ defined by $f = m \circ (p \times p)$, which fits along the bottom of the following diagram:



Since $L \times L$ is simply connected the lift $\tilde{f}$ of f (starting at the basepoint $e' \in L$) exists and is unique by Corollary 1.2. Now define multiplication in L as follows: for $n, k \in L$ define $n * k = \tilde{f}(n, k)$. We will show that this multiplication satisfies the group axioms.

Associativity: Consider the following diagram:



There again exists a unique lift preserving basepoints since $L \times L \times L$ is simply connected. Consider the maps $\tilde{g}, \tilde{h} : L \times L \times L \to L$ defined by $\tilde{g}(n, k, j) = \tilde{f}(n, \tilde{f}(k, j)) = n * (k * j)$ and $\tilde{h}(n, k, j) = \tilde{f}(\tilde{f}(n, k), j) = (n * k) * j$. We will show that both $\tilde{g}$ and $\tilde{h}$ are lifts of the map $f \circ (1 \times \tilde{f})$ in the diagram above and, by uniqueness of lifts, are equal. This will prove associativity.

First we note that $f \circ (1 \times \tilde{f})(n, k, j) = m(p(n), f(k, j)) = p(n)p(k)p(j)$. Now, to check that $\tilde{g}$

is a lift, we show that $(p \circ \tilde{g})(n,k,j) = p(n)p(k)p(j)$, which can be seen as follows:

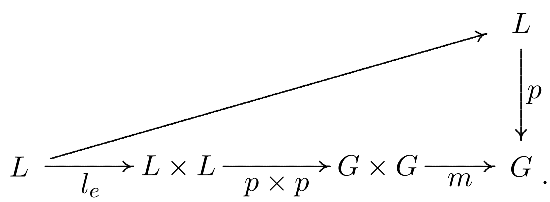
$$\begin{aligned} p \circ \tilde{g}(n,k,j) &= f(n, \tilde{f}(k,j)) \\ &= m(p(n), f(k,j)) \\ &= m(p(n), m(p(k), p(j))) \\ &= p(n)p(k)p(j). \end{aligned}$$

Similarly,

$$\begin{aligned} p \circ \tilde{h}(n,k,j) &= f(\tilde{f}(n,k), j) \\ &= m(f(n,k), p(j)) \\ &= m(m(p(n), p(k)), p(j)) \\ &= p(n)p(k)p(j), \end{aligned}$$

and so $\tilde{h}$ is a lift. Hence $\tilde{g} = \tilde{h}$ and the operation is associative.

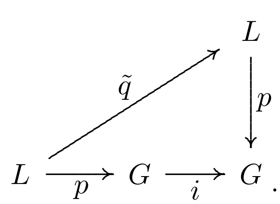
Identity: We wish to show that $e' \in L$ is the identity, i.e. $\tilde{f}(e', n) = \tilde{f}(n, e') = n$ for all $n \in L$. Define a map $l_e : L \to L \times L$ by $l_e(n) = (e', n)$; it is clear that l_e is continuous. We then have the following diagram:



We again know a lift exists since L is simply connected. We wish to show that $\tilde{f} \circ l_e(n) = \tilde{f}(e', n) = e' * n = n$ for all $n \in L$.

By similar reasoning as before, we will show that both $\tilde{f} \circ l_e$ and the identity map are lifts (starting at $e' \in L$) of the map $f \circ l_e$ and are hence the same. Along the bottom of the diagram we have that $f \circ l_e(n) = f(e', n) = m(p(e'), p(n)) = p(n)$, hence the identity map is a lift of $f \circ l_e$. Moreover, since $\tilde{f}$ is a lift of f, we have $p \circ \tilde{f} \circ l_e = f \circ l_e$. Hence $\tilde{f} \circ l_e$ is also a lift, and so this is equal to the identity map, as desired. Showing that $n * e' = n$ is similar and simply uses the map $r_e : L \to L \times L$ defined by $r_e(n) = (n, e')$ instead of l_e.

Inverses: We wish to show that for all $n \in L$ there is an element $n^{-1} \in L$ such that $n^{-1} * n = n * n^{-1} = e'$. Let i denote the inversion in G so that $i : G \to G$ by $i(g) = g^{-1}$ and let $q = i \circ p$. We then have the following diagram:



The lift $\tilde{q}$ exists for the usual reasons, L being simply connected. We will show that $\tilde{q}(n)$ is the inverse of n for all $n \in L$. To see this, consider the following diagram, where f is defined as before:

$$\begin{array}{ccccc} & & & & L \\ & & \nearrow & & \Big\downarrow p \\ L & \xrightarrow[\tilde{q} \times 1]{} & L \times L & \xrightarrow[f]{} & G \end{array}.$$

Consider the map $\tilde{j} : L \to L$ defined by $\tilde{j} = \tilde{f} \circ (\tilde{q} \times 1)$, noting that $\tilde{j}(n) = \tilde{q}(n) * n$. We will show that the constant map $e_L : L \to L$ to e' and $\tilde{j}$ are both lifts of $f \circ (\tilde{q} \times 1)$. This will show that $\tilde{q}(n) * n = e'$ for all $n \in L$.

First we have $f \circ (\tilde{q} \times 1)(n) = f(\tilde{q}(n), n) = m(q(n), p(n)) = m(p(n)^{-1}, p(n)) = e$. As this is a constant map, it is clear that e_L is a lift. As $\tilde{f}$ lifts f, we have $p \circ \tilde{j} = p \circ \tilde{f} \circ (\tilde{q} \times 1) = f \circ (\tilde{q} \times 1)$, so that $\tilde{j}$ is also a lift of $f \circ (\tilde{q} \times 1)$. Hence $\tilde{j}(n) = \tilde{q}(n) * n = e'$ as desired. Showing that $n * \tilde{q}(n) = e'$ follows an identical procedure.

Now we have that L is a group with operation $n * k = \tilde{f}(n, k)$. But because all maps in the above diagrams are continuous, we have that both $\tilde{f}$ and $\tilde{q}$ are continuous and hence the group operation and inversion maps are continuous. Thus L is a topological group. Further, we have

$$p(n * k) = p \circ \tilde{f}(n, k) = f(n, k) = m(p(n), p(k)) = p(n)p(k),$$

and so $p : L \to G$ is a homomorphism.

Uniqueness: We know that L is unique as a space since it is a universal cover. Let $g : L \times L \to L$ be an operation on L denoted by $g(n, k) = n\#k$. Suppose that this operation permits a topological group structure on L for which p is a homomorphism. We will show that for all $n, k \in G$ we have $n\#k = g(n, k) = \tilde{f}(n, k) = n * k$.

Recall that $f = m \circ (p \times p)$. We will show that g is a lift of f and by uniqueness of lifts it is therefore equal to $\tilde{f}$. To do this, we show that $p \circ g = f$. Note that

$$\begin{aligned} p \circ g(n, k) &= p(n\#k) \\ &= p(n)p(k) \\ &= m(p(n), p(k)) \\ &= f(n, k). \end{aligned}$$

Hence we have uniqueness of L as a topological group.

□

Note that in the above theorem, a different choice of point in the fiber yields an isomorphic group structure on L. Since $SO(n)$ is a topological group and $\mathrm{Spin}(n)$ is its universal cover, we immediately obtain the following corollary.

Corollary 3.4. *The spin groups are topological groups.*

We have now defined and demonstrated the existence of the spin groups. We even know that they are topological groups, but what can we say about them in detail? Incidentally, we have actually already encountered the first spin group.

Proposition 3.5. *The topological group Spin*(3) *is homeomorphic to* S^3.

Proof. As we have seen before in Proposition 2.11, the quotient map $q : S^3 \to \mathbb{R}P^3 \cong SO(3)$ is a covering map. Further, it is a cover of two sheets since for any point $r \in \mathbb{R}P^3$ we have that $q^{-1}(r)$ is a pair of antipodes. Thus S^3 is a double cover of $SO(3)$, so it is universal as well. But since the universal double cover is unique, we have $\text{Spin}(3) \cong S^3$.

□

We will see later that $\text{Spin}(3) \approx S^3$ as groups as well.

4 A Matrix Representation of Spin(3)

4.1 The Basics of the Quaternions

The quaternions constitute what is called a division ring, which is a ring that satisfies all of the axioms of a field except for commutativity of multiplication. The quaternions may be thought of as an extension of the complex numbers.

Definition 4.1. The algebra of **quaternions**, denoted $\mathbb{H}$, is a four-dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$ and vector multiplication generated by the equations $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Multiplication on $\mathbb{H}$ is given by distributing linearly over addition.

An element $q \in \mathbb{H}$ is expressed as a linear combination $q = a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$. From the definition of quaternionic multiplication we obtain the following rules for multiplying individual basis elements:

$$\begin{array}{rclcrcl} ij &=& k & \quad & ji &=& -k \\ jk &=& i & & kj &=& -i \\ ki &=& j & & ik &=& -j. \end{array}$$

For example, since $ijk = -1$, by multiplying both sides on the left by i we obtain $iijk = -i$. Since $i^2 = -1$ this simplifies to $jk = i$. Using these rules, multiplying quaternions becomes a simple, though possibly tedious task. Before we continue it is helpful to introduce a couple of definitions.

Definition 4.2. Let $q = a + bi + cj + dk$ be a quaternion. The **quaternionic conjugate** of q, denoted $\bar{q}$, is defined by $\bar{q} = a - bi - cj - dk$.

Definition 4.3. The **norm** is a map $\mathrm{N} : \mathbb{H} \to \mathbb{R}$ defined by $\mathrm{N}(q) = (q\bar{q})^{\frac{1}{2}}$. Put another way, given a quaternion $q = a+bi+cj+dk$, we have that the norm of q is given by $\mathrm{N}(q) = (a^2+b^2+c^2+d^2)^{\frac{1}{2}}$.

It follows immediately from the definition of the quaternionic conjugate and the rules for multiplication that the norm is multiplicative. That is, for $q, w \in \mathbb{H}$ we have $\mathrm{N}(qw) = \mathrm{N}(q)\mathrm{N}(w)$. Additionally, since the quaternions form a division ring, every nonzero element has an inverse, and conveniently there is even a formula for it. The inverse of a quaternion q is denoted q^{-1} and is given by $q^{-1} = \frac{\bar{q}}{(\mathrm{N}(q))^2}$. To check we simply compute:

$$qq^{-1} = \frac{q\bar{q}}{(\mathrm{N}(q))^2} = \frac{(\mathrm{N}(q))^2}{(\mathrm{N}(q))^2} = 1 = \frac{(\mathrm{N}(q))^2}{(\mathrm{N}(q))^2} = \frac{\bar{q}q}{(\mathrm{N}(q))^2} = q^{-1}q.$$

Proposition 4.1. *The set of unit length quaternions is a subgroup under multiplication.*

Proof. The set of unit length quaternions is closed under multiplication since, for any two elements q and w in it, we have $\mathrm{N}(q)\mathrm{N}(w) = \mathrm{N}(qw) = 1$. It is closed under inverses as well since, for any unit length quaternion q, we have $q^{-1} = \frac{\bar{q}}{(\mathrm{N}(q))^2} = \bar{q}$, which is still a unit length quaternion. Lastly, the identity is clearly a unit length quaternion, and so the set of all unit length quaternions is a subgroup.

□

Just as the complex plane is homeomorphic to $\mathbb{R}^2$, the quaternions are homeomorphic to Euclidean four-space $\mathbb{H} \cong \mathbb{R}^4$. This means that the group of unit length quaternions is homeomorphic to the three-sphere $S^3 \subseteq \mathbb{R}^4$. In general, spheres rarely admit the structure of a topological group, but here we see that the three-sphere in fact does.

4.2 The Basics of the Clifford Algebra

It is often stated without proof that the spin groups can be constructed as a subgroup of the invertible elements of the Clifford algebra. We will investigate the Clifford algebra and how it relates to both the spin groups and the quaternions. In this paper we will restrict ourselves to the Clifford algebra obtained from a three-dimensional vector space over the real numbers.

Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a basis for a vector subspace of M_4. An element in this vector space has the form $\mathbf{v} = v_1\mathbf{x}_1 + v_2\mathbf{x}_2 + v_3\mathbf{x}_3$ for $v_1, v_2, v_3 \in \mathbb{R}$. We may define a multiplication on this vector space by distributing over the addition and using matrix multiplication to multiply basis elements. That is, we define the product of two vectors $\mathbf{v}$ and $\mathbf{w}$ by

$$\begin{aligned}
\mathbf{vw} &= (v_1\mathbf{x}_1 + v_2\mathbf{x}_2 + v_3\mathbf{x}_3)(w_1\mathbf{x}_1 + w_2\mathbf{x}_2 + w_3\mathbf{x}_3) \\
&= v_1w_1\mathbf{x}_1\mathbf{x}_1 + v_1w_2\mathbf{x}_1\mathbf{x}_2 + v_1w_3\mathbf{x}_1\mathbf{x}_3 + \\
&\quad v_2w_1\mathbf{x}_2\mathbf{x}_1 + v_2w_2\mathbf{x}_2\mathbf{x}_2 + v_2w_3\mathbf{x}_2\mathbf{x}_3 + \\
&\quad v_3w_1\mathbf{x}_3\mathbf{x}_1 + v_3w_2\mathbf{x}_3\mathbf{x}_2 + v_3w_3\mathbf{x}_3\mathbf{x}_3.
\end{aligned}$$

The vector space spanned by $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is not necessarily closed under this multiplication. However, suppose that the matrices chosen to represent the basis elements satisfy the following two equations:

$$\begin{array}{lll}
(1) & \mathbf{x}_i^2 = I & \text{for } i = 1, 2, 3 \\
(2) & \mathbf{x}_i\mathbf{x}_j = -\mathbf{x}_j\mathbf{x}_i & \text{for } i \neq j.
\end{array}$$

We may obtain an eight-dimensional vector space by taking as its basis the set of all linearly independent products of $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$. By doing this, the vector space obtained will certainly be closed under the multiplication mentioned above. Upon taking every product, we find this set to be $\{I, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1\mathbf{x}_2, \mathbf{x}_2\mathbf{x}_3, \mathbf{x}_3\mathbf{x}_1, \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\}$. It contains only these eight matrices, as every other product reduces to a linear combination of these eight by equations (1) and (2).

Example 4.1. Consider the product $\mathbf{x}_1\mathbf{x}_3\mathbf{x}_2\mathbf{x}_3$. It reduces as follows:

$$\begin{aligned}
\mathbf{x}_1\mathbf{x}_3\mathbf{x}_2\mathbf{x}_3 &= (\mathbf{x}_1\mathbf{x}_3)(\mathbf{x}_2\mathbf{x}_3) \\
&= (\mathbf{x}_1\mathbf{x}_3)(-\mathbf{x}_3\mathbf{x}_2) \\
&= -\mathbf{x}_1(\mathbf{x}_3\mathbf{x}_3)\mathbf{x}_2 \\
&= -\mathbf{x}_1(I)\mathbf{x}_2 \\
&= -\mathbf{x}_1\mathbf{x}_2.
\end{aligned}$$

Similarly, every other possible product reduces to a linear combination of the eight above. For notational convenience, we write

$$\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_{12}, \quad \mathbf{x}_2\mathbf{x}_3 = \mathbf{x}_{23}, \quad \mathbf{x}_3\mathbf{x}_1 = \mathbf{x}_{31}, \quad \text{and} \quad \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 = \mathbf{x}_{123}.$$

This eight-dimensional vector space spanned by $\{I, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{31}, \mathbf{x}_{123}\}$ has essentially been constructed to "force" the multiplication to be closed. The vector space with this multiplication is called the *Clifford algebra*. It should be noted that there actually exist matrices to represent $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ that satisfy equations (1) and (2). Such matrices are given by John Snygg in [5] as

$$\mathbf{x}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The reader may check that these matrices do in fact satisfy the required equations. Given these three, we may take their products to deduce the rest of the matrices in the basis of our Clifford algebra. They are

$$\mathbf{x}_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \mathbf{x}_{23} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{x}_{31} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{x}_{123} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and I, which is of course the identity matrix.

There are in fact other Clifford algebras, and given any finite-dimensional vector space V it is possible to construct a Clifford algebra from it. There is a certain degree of freedom when doing so, as one may elect to alter equation (1) so that some basis elements $\mathbf{x}_j \in V$ satisfy $\mathbf{x}_j^2 = -I$ instead of $\mathbf{x}_j^2 = I$. For example, using the same techniques that we have employed, one may construct a 16-dimensional Clifford algebra from a four-dimensional Euclidean vector space. However, we will not investigate such algebras in this paper.

4.3 Relating Spin(3), the Quaternions, and the Clifford Algebra

If we look at the basis elements of the Clifford algebra from the previous section, we notice that many of them share some similar properties with the basis elements of the quaternions. In fact we have that $\mathbf{x}_{12}^2 = \mathbf{x}_{23}^2 = \mathbf{x}_{31}^2 = -I$ since, for example, $\mathbf{x}_{12}^2 = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2 = -\mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_1 = -\mathbf{x}_1 I \mathbf{x}_1 = -I$. This is the same as the equation $i^2 = j^2 = k^2 = -1$ for the quaternions. However, we have $\mathbf{x}_{12}\mathbf{x}_{23}\mathbf{x}_{31} = I$, which does not resemble the quaternion equation $ijk = -1$.

We would like $\mathbf{x}_{12}$ to play the role of i, $\mathbf{x}_{23}$ the role of j, and $\mathbf{x}_{31}$ the role of k, but we have just seen that they do not satisfy the quaternion equations. However, if instead of $\mathbf{x}_{31}$ we choose to use $-\mathbf{x}_3\mathbf{x}_1 = \mathbf{x}_1\mathbf{x}_3$, which we will write as $\mathbf{x}_{13}$, then the first equation still holds since its square is still $-I$. The second equation now appears as $\mathbf{x}_{12}\mathbf{x}_{23}\mathbf{x}_{13} = \mathbf{x}_1(I)\mathbf{x}_3\mathbf{x}_1\mathbf{x}_3 = \mathbf{x}_{13}^2 = -I$. This is now the same as the quaternion equation $ijk = -1$. Hence we have that the subspace of the

Clifford algebra given by the basis $\{I, \mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{13}\}$ is isomorphic to $\mathbb{H}$. In other words, the Clifford algebra we obtained from a three-dimensional vector space actually contains the quaternions as a subalgebra. Using this, we will now realize S^3 as a multiplicative subgroup of the Clifford algebra.

Now, since we have actual matrices to represent these basis elements, we can write an arbitrary quaternion q as a general linear combination, $q = aI + b\mathbf{x}_{12} + c\mathbf{x}_{23} + d\mathbf{x}_{13}$. Upon evaluating this with the given matrices we obtain

$$q = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Further, since we know that the set of unit length quaternions is homeomorphic to S^3, which is homeomorphic to Spin(3), we have that the space of unit length quaternions is homeomorphic to Spin(3). Recall that S^3 is a topological group. However, at this point we cannot be sure whether the topological group structure of S^3 as a multiplicative subgroup of the Clifford algebra is the same as the structure of the topological group Spin(3). By considering the covering map from S^3 to $SO(3)$ we shall see that they are in fact the same.

Theorem 4.2. *The topological group Spin(3) is homeomorphic and isomorphic to S^3.*

Proof. Let $q \in S^3$ and $v \in \mathbb{R}^4$. Thinking of q and v as quaternions, let R_q be the rotation that maps v to qvq^{-1} (where the multiplication and inversion are quaternionic). In Chapter 3.D of [2] Allen Hatcher claims that the map $p : S^3 \to SO(3)$ defined by $p(q) = R_q$ is a covering map. This can be seen by identifying $SO(3)$ with $\mathbb{R}P^3$, whereupon p is the natural antipodal cover of $\mathbb{R}P^3$. This map certainly sends antipodes to the same point since $(-q)v(-q)^{-1} = qvq^{-1}$, and so R_q and R_{-q} are the same rotation.

We now check that $p : S^3 \to SO(3)$ is a homomorphism. For $q, u \in S^3$ we have that $p(qu)$ is the rotation that maps $v \in \mathbb{R}^4$ to $(qu)v(qu)^{-1}$. But

$$(qu)v(qu)^{-1} = (qu)v(u^{-1}q^{-1}) = q(uvu^{-1})q^{-1} = R_q(R_u(v)),$$

which rotates v by R_u and then by R_q. Hence $p(qu) = R_qR_u = p(q)p(u)$. Additionally, Theorem 3.3 guarantees us that Spin(3) is the unique topological group that double covers $SO(3)$ by a homomorphism. As we have shown that S^3 has the same properties, we now have that Spin(3) is both homeomorphic and isomorphic to S^3.

$\square$

In short, we have found that Spin(3) is both homeomorphic and isomorphic to the group of unit length quaternions, which constitutes a multiplicative subgroup of the Clifford algebra. In order to obtain concrete matrix representations of the elements of Spin(3) we now need only do so for the unit length quaternions. Above we have already done so for general quaternions, and so by taking those matrices that satisfy $a^2 + b^2 + c^2 + d^2 = 1$, we have matrix representations for unit length quaternions, and hence elements of Spin(3).

Now we may directly express the covering map $p : S^3 \to SO(3)$ in terms of our quaternionic representations of elements of Spin(3). Given $q \in$ Spin(3) we may write it as a unit length quaternion $q = a + bi + cj + dk$. It maps to the rotation R_q defined by $R_q(v) = qvq^{-1}$. By computing the matrix representing this linear transformation we obtain the matrix representation of $p(q) \in SO(3)$. In this way, for $q \in \text{Spin}(3)$ we have

$$p(q) = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+db) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{pmatrix},$$

where of course $a^2+b^2+c^2+d^2=1$. To verify that $p(q) \in SO(3)$ we first take its determinant, which we find to be $(a^2+b^2+c^2+d^2)^3 = 1$. Second, if we let $p(q) = A$ we find

$$AA^T = A^TA = \begin{pmatrix} (a^2+b^2+c^2+d^2)^2 & 0 & 0 \\ 0 & (a^2+b^2+c^2+d^2)^2 & 0 \\ 0 & 0 & (a^2+b^2+c^2+d^2)^2 \end{pmatrix} = I.$$

Hence we have $p(q) \in SO(3)$.

4.4 Conclusion

In summary, starting with the problem of 4π-periodicity we provide a review of homotopy theory and covering spaces. We then focus our attention on the special orthogonal groups, and after analyzing these groups of rotations, we use some bundle theory to produce a long exact sequence of groups and homomorphisms. From this, we compute $\pi_1(SO(n))$. This calculation provides an explanation and model for the 4π-periodicity phenomenon, but is only enough to demonstrate the existence of the spin groups. In order to further evaluate the spin groups we need more. By introducing the quaternions and some Clifford algebra, we complete an analysis of Spin(3) and produce concrete matrix representations of its elements. We conclude by providing the cover that directly maps each element of Spin(3) down to a matrix in $SO(3)$.

The question of why the 4π-periodicity phenomenon exists and how it works turns out to be a topological one. The answer is rooted in the special orthogonal groups, and it is the fundamental group that detects it.

A Appendix

In order to prove Theorem 3.1 we must first have a lemma that allows us to lift homotopies.

Lemma A.1. *Let* $F \xrightarrow{i} E \xrightarrow{p} B$ *be a fiber sequence, and suppose we have the following commutative diagram:*

$$\begin{array}{ccc} & & E \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ I^n & \xrightarrow{f} & B \end{array}.$$

Suppose $H : I^n \times I \to B$ *is a homotopy such that* $H(-,0) = f$ *and* $H(-,t)$ *maps the boundary of* I^n *to the basepoint for all* $t \in I$*. Then there exists a homotopy* $\tilde{H} : I^n \times I \to E$ *with* $\tilde{H}(-,0) = \tilde{f}$ *that satisfies* $p \circ \tilde{H} = H$ *such that* $\tilde{H}(-,t)$ *also maps the boundary of* I^n *to the basepoint for all* t*.*

The proof runs exactly as it does for lifts in covering spaces. To see a proof we refer the reader to Theorem 11.7 of [6].

Using this lemma we may now proceed with the proof of Theorem 3.1, which consists of producing the homomorphism $\partial : \pi_n(B) \to \pi_{n-1}(F)$ and showing the six set inclusions that make the desired sequence exact. We restate the theorem and then proceed with the proof.

Theorem 3.1. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fiber sequence, then there exists a homomorphism $\partial : \pi_n(B) \to \pi_{n-1}(F)$ such that the following sequence is exact:

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{i_*} \pi_{n-1}(E) \longrightarrow \cdots.$$

Proof. First we must construct the homomorphism $\partial : \pi_n(B) \to \pi_{n-1}(F)$. Let $\beta : I^n \to B$ be a loop in B. Because it is a loop it maps the boundary of I^n to the basepoint b_0. We may think of this loop as a homotopy $\beta : I^{n-1} \times I \to B$ starting at $\beta|_{I^{n-1}\times\{0\}}$. Now by the previous lemma we know that β has a lift $\tilde{\beta} : I^{n-1} \times I \to E$ with $\tilde{\beta}|_{I^{n-1}\times\{0\}} = e_0$. Hence $\tilde{\beta}|_{I^{n-1}\times\{1\}} : I^{n-1} \to F$ and so we define $\partial([\beta]) = [\tilde{\beta}|_{I^{n-1}\times\{1\}}]$.

This is a well defined map from $\pi_n(B)$ to $\pi_{n-1}(F)$ since the homotopy between any two homotopic loops in B also lifts. In addition, by proper choice of lifts we have

$$\partial([\alpha] * [\beta]) = \partial([\alpha * \beta]) = [\widetilde{\alpha * \beta}|_{I^{n-1}\times\{1\}}] = [\tilde{\alpha} * \tilde{\beta}|_{I^{n-1}\times\{1\}}] = \partial([\alpha]) * \partial([\beta]).$$

Hence ∂ is a homomorphism. Next we show exactness.

As our first step towards exactness we show $\mathrm{im}(i_*) = \ker(p_*)$. Let $[g] \in \mathrm{im}(i_*)$. There exists $[f] \in \pi_n(F)$ such that $i_*([f]) = [i \circ f] = [g]$. Thus we have $p_*([g]) = [p \circ g] = [p \circ i \circ f] = (p \circ i)_*([f]) = 0$ since $p \circ i$ is a constant map. Hence $[g] \in \ker(p_*)$ and we have $\mathrm{im}(i_*) \subseteq \ker(p_*)$.

Now let $[f] \in \ker(p_*)$. We have $p_*([f]) = 0$, i.e. $[p \circ f] = [b_0]$, which means $p \circ f \simeq b_0$. Hence there exists a homotopy $H : I^n \times I \to B$ so that $H(-,0) = p \circ f$ and $H(-,1) = b_0$. Lift H to a homotopy $\tilde{H}$ between loops in E so that $\tilde{H}(-,0) = f$. Now the map $g = \tilde{H}(-,1) : I^n \to F$ is a loop in F. We have $i_*([g]) = [i \circ g]$, but since $\tilde{H}$ is a homotopy between f and g we have $i_*([g]) = [f]$ and so $\ker(p_*) \subseteq \mathrm{im}(i_*)$. Thus we have $\mathrm{im}(i_*) = \ker(p_*)$.

Next we show $\mathrm{im}(p_*) = \ker(\partial)$. Let $[g] \in \mathrm{im}(p_*)$. There exists $[f] \in \pi_n(E)$ such that $p_*([f]) = [p \circ f] = [g]$ and so $p \circ f \simeq g$. Lifting $p \circ f$ so that $\widetilde{p \circ f}(-, 0) = e_0$ we see that f itself is such a lift. Thus we have $\partial([p \circ f]) = [f|_{I^{n-1} \times \{1\}}] = [e_0]$, and so $\mathrm{im}(p_*) \subseteq \ker(\partial)$.

Let $[f] \in \ker(\partial)$. We have $\partial([f]) = [\tilde{f}|_{I^{n-1} \times \{1\}}] = [e_0]$. There exists a homotopy $H : I^{n-1} \times I \to F$ such that $H(-, 0) = \tilde{f}|_{I^{n-1} \times \{1\}}$ and $H(-, 1) = e_0$. However, $\tilde{f} : I^{n-1} \times I \to E$ may also be thought of as a homotopy, and so by pasting the two together we form the homotopy $\alpha : I^{n-1} \times I \to E$ given by

$$\alpha(-, t) = \begin{cases} \tilde{f}(-, 2t), & t \in [0, \frac{1}{2}] \\ (i \circ H)(-, 2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

The two homotopies agree at $t = \frac{1}{2}$ so this combination is continuous by the pasting lemma. Further, the combination is a loop in E, and $p_*([\alpha]) = [p \circ \alpha] = [f]$. Hence we have $\ker(\partial) \subseteq \mathrm{im}(p_*)$, and so $\mathrm{im}(p_*) = \ker(\partial)$.

Lastly we show that $\mathrm{im}(\partial) = \ker(i_*)$. Let $[g] \in \mathrm{im}(\partial)$. There exists $[f] \in \pi_n(B)$ such that $\partial([f]) = [g]$. We have $i_*(\partial([f])) = [i \circ \partial([f])] = [i \circ \tilde{f}|_{I^{n-1} \times \{1\}}]$. We want $i \circ \tilde{f}|_{I^{n-1} \times \{1\}} \simeq e_0$. We have that $\tilde{f}$ is a homotopy in E with $\tilde{f}(-, 1) = \partial([f])$ and $\tilde{f}(-, 0) = e_0$. Hence $i \circ \tilde{f}|_{I^{n-1} \times \{1\}} \simeq e_0$ and so $\mathrm{im}(\partial) \subseteq \ker(i_*)$.

Let $[f] \in \ker(i_*)$. We have $i_*([f]) = [i \circ f] = [e_0]$. There exists a homotopy $H : I^n \times I \to E$ such that $H(-, 0) = e_0$ and $H(-, 1) = i \circ f$. Let $\alpha = p \circ H : I^{n+1} \to B$. Now H is a lift of α, which is a loop in B. We have $\partial([\alpha]) = [\tilde{\alpha}|_{I^n \times \{1\}}] = [H|_{I^n \times \{1\}}] = [f]$, and hence $\ker(i_*) \subseteq \mathrm{im}(\partial)$. Thus we have $\mathrm{im}(\partial) = \ker(i_*)$, which completes the proof.

□

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