A Lie-Theoretic Approach to the Spin Groups

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Abstract

We investigate a collection of topological groups known as spin groups from a Lie-theoretic viewpoint. From this analysis, we compute well-known groups isomorphic to both Spin(3) and Spin(4), and generate matrix representations of their elements.

1 Introduction

The purpose of this paper is to provide an analysis of the spin groups through an alternative viewpoint to that taken in [2]. In addition to verifying our previous results regarding Spin(3) through new means, we will produce real matrix representations for the higher case of Spin(4) as well. The approach taken will be Lie-theoretic and will view the groups under consideration as manifolds. In [2] we do not arrive at matrices until near the end, when we are forced to seek matrices from the Clifford algebra in order to represent elements of Spin(3). However, by analyzing the spin groups from the perspective of manifold theory, we will be afforded easy access to matrices at the start of our efforts.

In this paper we aim similarly to find matrix representations for the elements of Spin(3), but we go about it by means of a new approach. The spin group Spin(n) can be seen algebraically as a special kind of group equipped with a subgroup $\mathcal{S}$ of order two for which the quotient Spin(n)/$\mathcal{S}$ is the special orthogonal group $SO(n)$. In other words, there is a special surjective two-to-one homomorphism that maps each spin group to its respective special orthogonal group. It should be noted, however, that finding Spin(n) is not as simple as trying such groups as $SO(n) \times \mathbb{Z}_2$ as the spin group must be connected. This is because of the special characteristics that the quotient map must have. We will see later that for our case, it comes down to the additional requirement that the map be a local homeomorphism.

To accomplish this task of uncovering spin groups, we employ Lie theory so that we may exploit the Inverse Function Theorem. The general paradigm goes as follows. Suppose we have a likely candidate X for the spin group Spin(n) along with a possible quotient map $p : X \to SO(n)$. Because of the uniqueness of such maps and spin groups, if we can show that p is a surjective, two-to-one homomorphism and local homeomorphism, then we know that $X \approx \text{Spin}(n)$ as a group and a space. Lie theory allows us to do this by checking that p is a local diffeomorphism at the identity. From the map $p : X \to SO(n)$, we obtain the new map $dp_I : L(X) \to L(SO(n))$ of the Lie algebras, which is the derivative of p evaluated at the identity. If dp_I sends a basis of $L(X)$ to a basis of $L(SO(n))$, then by the Inverse Function Theorem p is a local diffeomorphism and hence a local homeomorphism. This assures us that we have found Spin(n).

We will begin with basic definitions of manifold theory, and will move from here into Lie algebra and its relation to covering spaces. With this point of view, we will have matrices available from the start, as matrix groups are in fact manifolds. We will then set about to compute Spin(4) through the same means.

2 Manifolds and Lie Algebra

In this section we will run through much of the material needed to build up a working knowledge of Lie theory and apply it to the task of computing spin groups. Most of the material comes from

Kristopher Tapp's book on matrix groups and will be explained here rather than proved, as it can be found in [4].

2.1 Basics of Manifolds

We begin by stipulating what we mean by *matrix group.* We will restrict this terminology to mean those groups of matrices that are closed in GL_n as subspaces. The entries of matrices of these groups may be real, complex, or quaternionic. For the case of real-valued matrices, we may topologize GL_n as a subspace of $\mathbb{R}^{n^2}$ by thinking of matrix entries as coordinates in this Euclidean space. Complex and quaternionic counterparts are topologized similarly in higher-dimensional Euclidean spaces. In fact, with this topology, the general linear group is a topological group, and so too are all matrix groups.

Definition 2.1. Let $U \subseteq \mathbb{R}^m$ be an open set. A map $f : U \to \mathbb{R}^n$ is said to be **smooth on U** if its rth order partial derivative exists and is continuous on U for all $r \in \mathbb{N}$. Suppose $f : X \to \mathbb{R}^n$ for an arbitrary subset $X \subseteq \mathbb{R}^m$. We simply call f **smooth** if for all $x \in X$ there is some neighborhood $V_x \subseteq \mathbb{R}^m$ of x equipped with some map $f' : V_x \to \mathbb{R}^n$ which is *smooth on* V_x such that f and f' agree on $X \cap V_x$.

Definition 2.2. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. A **diffeomorphism** is a smooth bijection $f : X \to Y$ whose inverse is also smooth. If $f : X \to Y$ is a diffeomorphism, then X and Y are said to be **diffeomorphic**.

It should be noted that every diffeomorphism is automatically a homeomorphism, making diffeomorphism a stronger condition. Speaking loosely, it has the additional condition of "continued differentiability." With this terminology, we turn our attention to manifolds. For our purposes we will take manifolds as being certain subsets of Euclidean spaces without viewing them from a purely intrinsic point of view. In this spirit we define a *manifold* M as a subset of $\mathbb{R}^m$ such that for all $p \in M$ there is a neighborhood V_p of p that is diffeomorphic to some open set $U \subseteq \mathbb{R}^n$. In this case, we say that the manifold M has dimension n.

Because manifolds are locally Euclidean in a differentiable manner, it makes sense to talk about the derivative of a map between manifolds. As we shall see however, this only makes sense once we specify a particular point within the manifold. We first consider the generalized derivative for Euclidean spaces. For a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$, the *derivative* is denoted *df* and is given by the Jacobian matrix of partial derivatives

$$df = \begin{pmatrix} \frac{\partial f^1}{\partial u^1} & \frac{\partial f^1}{\partial u^2} & \cdots & \frac{\partial f^1}{\partial u^m} \\ \frac{\partial f^2}{\partial u^1} & \frac{\partial f^2}{\partial u^2} & \cdots & \frac{\partial f^2}{\partial u^m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial u^1} & \frac{\partial f^n}{\partial u^2} & \cdots & \frac{\partial f^n}{\partial u^m} \end{pmatrix}.$$

Once the derivative is evaluated at a point $p \in \mathbb{R}^m$, the entries become real numbers and the matrix represents a linear map.

We now have a clear way of discussing the derivative of a map between Euclidean spaces. Since every point in a manifold lies in a neighborhood that is diffeomorphic to some open Euclidean subspace, this leads to a method for treating derivatives of maps between manifolds.

We treat the derivative of such a map at a point within some Euclidean neighborhood in a way purely analogous to our Euclidean treatment. However, we cannot speak solely about the derivative of a map between manifolds without first nailing it down at a point. This is because without doing so, we do not know which local neighborhood to use in our Euclidean treatment of the derivative. Further, we cannot continue without this information because the individual entry functions of the Jacobian matrix are liable to change abruptly to entirely different functions as we change Euclidean neighborhoods in the manifold.

Suppose $f : M_1 \to M_2$ is a smooth map between manifolds M_1 and M_2, which have dimension m and n respectively. To make sense of the derivative of f inside some Euclidean neighborhood V_p of $p \in M_1$, what we are doing is moving through several steps. First we find a Euclidean neighborhood $V_{f(p)}$ of $f(p) \in M_2$, compatible with f. Next we move through the diffeomorphism ϕ_1 that sends points in V_p to an open subset U_1 of $\mathbb{R}^m$. Next we move to an open subset U_2 of $\mathbb{R}^n$ and lastly back through the inverse diffeomorphism of U_2 into the Euclidean neighborhood $V_{f(p)}$. It looks something like this:

$$\begin{array}{ccc} M_1 \supseteq V_p & \xrightarrow{\quad f \quad} & V_{f(p)} \subseteq M_2 \\ \phi_1^{-1} \uparrow \downarrow \phi_1 & & \phi_2^{-1} \uparrow \downarrow \phi_2 \\ \mathbb{R}^m \supseteq U_1 & \xrightarrow{\quad \tilde{f} \quad} & U_2 \subseteq \mathbb{R}^n \end{array}$$

where each ϕ_i is a diffeomorphism between the appropriate Euclidean neighborhood of M_i and U_i. In this picture, for all $x \in V_p$ the map $\tilde{f}$ sends $\phi_1(x)$ to $\phi_2(f(x))$ so that the diagram commutes. Because each ϕ_i is a diffeomorphism and $\tilde{f}$ is a map between purely Euclidean spaces, we may understand the derivative of f at p as the derivative of $\tilde{f}$. We write this as df_p, which is a linear map $\mathbb{R}^m \to \mathbb{R}^n$.

Suppose there exists a sequence of smooth maps between manifolds

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 .$$

It turns out that the chain rule from multivariable calculus holds when extended to manifolds. This is a simple consequence of the locally Euclidean nature of manifolds. We state this here as a proposition.

Proposition 2.1 (The Chain Rule)**.** *If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are smooth maps between manifolds, then $d(g \circ f)_p = dg_{f(p)} \circ df_p$.*

The chain rule is a powerful tool and will return later to provide an important result. In the meantime, our ultimate reason for discussing manifold theory is to find spin groups. It is the close relation between matrix groups and manifolds that helps us accomplish this. As can be seen formally in § 7 of [4], it turns out that matrix groups are in fact manifolds.

2.2 Manifolds with Group Structure

The first idea in relation to Lie algebra is perhaps the most intuitive, and that is the concept of the tangent space to a point on a manifold. It is a natural extension of tangent lines and tangent planes from calculus. In order to construct this we will make use of paths in matrix groups. To differentiate a path in a matrix group, one simply differentiates the individual entries.

Definition 2.3. Let G be a matrix group and let p be a point in G. Let Γ be the set of all differentiable paths $\gamma : (-\varepsilon, \varepsilon) \to G$ such that $\gamma(0) = p$. The set consisting of $\frac{d}{dt}\big|_{t=0}\gamma(t)$ for all $\gamma \in \Gamma$ is called the **tangent space** of G at the point p, which we denote by T_pG.

In following our previous notation we may write $\frac{d}{dt}\big|_{t=0}\gamma(t)$ as $d\gamma_0$. Each such $d\gamma_0$ is the derivative of a path through p evaluated at time zero and constitutes a vector in the tangent space. The tangent space is a vector space with dimension the same as its matrix group.

Example 2.1. Consider the matrix group

$$S^1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}.$$

This is the familiar unit circle, whose dimension is of course one. Let p be the point $(1, 0) \in S^1$. Visually, we may think of the tangent space at p as the vertical line that touches S^1 at p. However, viewed as a subspace of the complex plane, the tangent space $T_pS^1 \approx \mathbb{R}$ is the imaginary axis. This will be of particular relevance later, as we shall similarly see that the tangent space of the 3-sphere at the identity is the imaginary portion of the quaternions, i.e. $\text{Span}\{i, j, k\}$.

The derivative allows us to construct vector spaces out of matrix groups by considering their tangent spaces at various points. Further, the derivative of a smooth map between matrix groups $f : G_1 \to G_2$ evaluated at a point is an $m \times n$ matrix, where n is the dimension of the group mapped out of (the domain), and m is the dimension of the group mapped into (the codomain). The tangent spaces of these groups will have respective dimensions n and m as well, hence the derivative acts as a linear map between these tangent spaces. In other words, if f is smooth, then we have

$$f : G_1 \to G_2 \implies df_p : T_pG_1 \to T_{f(p)}G_2.$$

In this sense, the derivative is a sort of local linearization of a map between manifolds. The derivative at a point linearizes not only the map within the point's neighborhood, but also the neighborhood itself in the form of the tangent space at that point. The tangent space at the identity of a matrix group is of particular importance. It is this vector space that will be made into the Lie algebra of the matrix group.

Definition 2.4. Let G be a matrix group and fix $g \in G$. Let $C_g : G \to G$ be the map defined by $C_g(x) = gxg^{-1}$. We define the **adjoint map** Ad_g as being the derivative of C_g at I so that $Ad_g : T_IG \to T_IG$.

We may compute the adjoint map explicitly as follows. If $B \in T_IG$, then $B = \frac{d}{dt}\big|_{t=0}b(t) = db_0$ for some differentiable path $b : (-\varepsilon, \varepsilon) \to G$ such that $b(0) = I$. Hence, by the chain rule we have

$$\begin{aligned} Ad_g(B) &= d(C_g)_I \circ db_0 \\ &= \left(d(C_g)_{b(t)} \circ db_t\right)\Big|_{t=0} \\ &= \frac{d}{dt}\Big|_{t=0} C_g(b(t)) \\ &= \frac{d}{dt}\Big|_{t=0} \left(gb(t)g^{-1}\right) \\ &= gBg^{-1}. \end{aligned}$$

In short, $Ad_g(B) = gBg^{-1}$. This leads to the formulation of a multiplication on T_IG, which is called the Lie bracket.

Definition 2.5. Let G be a matrix group and let $A, B \in T_IG$. Suppose $a : (-\varepsilon, \varepsilon) \to G$ is some differentiable path such that $a(0) = I$ and $da_0 = A$. Then the **Lie bracket** $[A, B]$ is defined by $[A, B] = \frac{d}{dt}\big|_{t=0} Ad_{a(t)}B$.

From this definition it is clear that the tangent space T_IG is closed under the Lie bracket operation. Precisely, $[A, B]$ is the initial velocity vector of a path in the Euclidean vector space T_IG, and hence $[A, B]$ remains in this vector space. While this definition is useful for many technical details, it can be simplified to a form which will be helpful to use in practice. The equivalent definition is obtained as follows:

$$\begin{aligned} [A, B] &= \frac{d}{dt}\Big|_{t=0} Ad_{a(t)}B \\ &= \frac{d}{dt}\Big|_{t=0} \big(a(t)Ba(t)^{-1}\big) \\ &= da_0Ba(0)^{-1} - a(0)Ba(0)^{-2}da_0 \\ &= AB - BA, \end{aligned}$$

where we have made use of the product rule. Hence $[A, B] = AB - BA$, which shows that the Lie bracket measures the failure of commutativity of matrix multiplication. The Lie bracket itself is anticommutative so that $[A, B] = -[B, A]$. We now have a reasonable multiplication on T_IG and are ready to define the Lie algebra.

Definition 2.6. Let G be a matrix group. The **Lie algebra** of G, denoted $L(G)$, is the vector space T_IG with the Lie bracket as multiplication.

It should be noted that the Lie algebra of G is not an associative algebra since

$$[A, [B, C]] = ABC - ACB - BCA + CBA$$

while

$$[[A, B], C] = ABC - BAC - CAB + CBA.$$

In fact,

$$[A, [B, C]] - [[A, B], C] = [B, [A, C]].$$

For our purposes, maps of interest between matrix groups are smooth homomorphisms. In this light, all maps between matrix groups will be assumed to be smooth homomorphisms from this point on, unless noted otherwise.

Definition 2.7. A **Lie map** is a linear map $f : L(G_1) \to L(G_2)$ between Lie algebras that preserves the Lie bracket operation. That is, $f([A, B]) = [f(A), f(B)]$.

It turns out that differentiation at the identity has some special aspects. Namely, differentiation of a smooth homomorphism between matrix groups evaluated at the identity acts as a construction of a Lie map between Lie algebras. Given a smooth homomorphism $f : G_1 \to G_2$, we define $L(f) = df_I : L(G_1) \to L(G_2)$. This construction has the following properties:

1. For all matrix groups G, the map constructed from the identity map on G is itself the identity map on the Lie algebra constructed from G. In symbols, $L(1_G) = 1_{L(G)}$.

2. The map constructed from a composition of two smooth homomorphisms is equal to the composition of the two maps constructed from each smooth homomorphism. In symbols, $L(f \circ g) = L(f) \circ L(g)$.

In other words, this derivative construction preserves identity maps and compositions of maps. These properties are not difficult to show. The construction of the new Lie map $L(f) : L(G_1) \to L(G_2)$ where $L(f) = df_I$ and $L(G) = T_I G$ is known as the *Lie functor*.

We first check that differentiation preserves identities. Let 1_G be the identity on the n-dimensional matrix group G. Decomposing 1_G into n separate functions f^i, we have that $\frac{\partial f^i}{\partial u^j} = \delta_{ij}$. Hence we have $L(1_G) = I$, which is the identity map on $L(G)$.

Next we verify that the derivative at the identity preserves composition. This fact is equivalent to the chain rule. Let $g : G_1 \to G_2$ and $f : G_2 \to G_3$ be smooth homomorphisms between matrix groups. By the chain rule we have that

$$L(f \circ g) = d(f \circ g)_I = df_{g(I)} \circ dg_I = L(f) \circ L(g),$$

where we use that g is a homomorphism to assure $g(I) = I$. Hence composition is preserved.

2.3 Matrix Groups and Useful Tools

We are now in an excellent position to begin discussing some particular matrix groups. We begin with some simple definitions and continue on to the familiar orthogonal group and its extension to groups of complex- and quaternionic-valued matrices. From now on we adopt the notation that if $\mathbb{K}$ is some real division algebra, i.e. a ring for which every nonzero element has a multiplicative inverse, then a matrix group $G(\mathbb{K})$ has entries in $\mathbb{K}$. We will also denote the division algebra of quaternions by $\mathbb{H}$.

Suppose q and w are elements of $\mathbb{R}^n$, $\mathbb{C}^n$, or $\mathbb{H}^n$. We define the *inner product* of q and w as being $\langle q, w \rangle = \sum_{i=1}^{n} q_i \bar{w}_i$. For the case of real-valued vectors, this is the simple dot product. The *hermitian conjugate* of a matrix A is the matrix $A^* = (\bar{A})^T$, obtained by taking the conjugate of the entries and then taking the transpose. Let

$$\mathcal{O}_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K}) \mid AA^* = A^*A = I\}.$$

It is easy to show that this is a subgroup of $GL_n(\mathbb{K})$. If $A \in \mathcal{O}_n(\mathbb{K})$ then $A^* = A^{-1} \in \mathcal{O}_n(\mathbb{K})$. Hence for $A, B \in \mathcal{O}_n(\mathbb{K})$ we have $AB(AB)^* = A(BB^*)A^* = I$, meaning $AB \in \mathcal{O}_n(\mathbb{K})$.

Definition 2.8. The group $\mathcal{O}_n(\mathbb{R})$ is called the **orthogonal group** and is denoted $O(n)$. The group $\mathcal{O}_n(\mathbb{C})$ is called the **unitary group** and is denoted $U(n)$. Lastly, the group $\mathcal{O}_n(\mathbb{H})$ is called the **symplectic group** and is denoted $\mathrm{Sp}(n)$.

Of particular importance to us is the group $\mathrm{Sp}(1)$. We already know that this is a matrix group as described above, but it turns out that it is also the three-sphere.

Proposition 2.2. *The matrix group* $\mathrm{Sp}(1)$ *is both isomorphic and homeomorphic to the topological group* S^3.

Proof. Since Sp(1) consists of 1×1 matrices, the transpose does nothing and we have that its elements are merely quaternions q that satisfy $q\bar{q} = 1$. This implies immediately that q is unit length and hence Sp(1) and S^3 are the same as sets. However, matrix multiplication of 1×1 matrices is merely multiplication of entries within the division algebra. Hence the group operations for Sp(1) and S^3 are both quaternionic multiplication and so they are isomorphic as groups. Lastly, both sit inside $\mathbb{H} \cong \mathbb{R}^4$ as the set of unit length quaternions and inherit the subspace topology, and so they are homeomorphic as spaces.

□

Since Sp(1) is homeomorphic to S^3, it has dimension three and hence its Lie algebra is isomorphic to $\mathbb{R}^3$ as a vector space. Further, in § 5 of [4] it is shown that $L(\mathrm{Sp}(1))$ is the span of the quaternionic basis elements $\{i, j, k\}$; see Example 2.1. In other words, the Lie bracket operation on this Lie algebra behaves just as quaternionic multiplication does for these elements. The same basis in the form of matrices would be $\{A, B, C\}$ such that $[A, B] = C$, $[B, C] = A$, and $[A, C] = -B$.

Another Lie algebra that will be important to us is $L(SO(n))$. It is also shown in § 5 of [4] that $L(SO(n)) = \{A \in M_n \mid A + A^T = 0\}$, the vector space of all skew-symmetric matrices. Knowing this for the case of $n = 3$ in terms of its basis is important enough for us to state here as a proposition.

Proposition 2.3. *Let the matrices E_1, E_2, and E_3 be given by*

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the Lie algebra of $SO(3)$ is the span of E_1, E_2, and E_3 under the Lie bracket operation.

The question now arises as to whether it is always possible to convert a quaternionic-valued or complex-valued matrix into an equivalent real-valued matrix. The answer is yes, and it is a fairly simple process to do so.

We begin by asking ourselves what it is that makes a matrix unique. In linear algebra, matrices represent linear transformations between vector spaces, and so if two matrices represent the same transformation they should be equivalent. An $n \times n$ real-valued matrix is a transformation from $\mathbb{R}^n$ to itself. Similarly, an $n \times n$ complex-valued matrix is a transformation from $\mathbb{C}^n$ to itself. Ignoring multiplication, we may make the identification $\mathbb{C} = \mathbb{R}^2$ as vector spaces. Accordingly, an $n \times n$ complex matrix is equivalent to a $2n \times 2n$ real matrix if they represent the same transformation from $\mathbb{R}^{2n}$ to itself.

Let us first define the *identification map* $\iota : \mathbb{C}^n \to \mathbb{R}^{2n}$ by the formula

$$\iota(a_1 + b_1 i, a_2 + b_2 i, \ldots, a_n + b_n i) = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n).$$

We now seek a map $\psi_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ such that for all $A \in M_n(\mathbb{C})$ and $\vec{v} \in \mathbb{C}^n$ we have $\iota(A\vec{v}) = \psi_n(A)\iota(\vec{v})$. For the case $n = 1$ we may easily compute this directly. Let $A = (a + bi)$ be an arbitrary 1×1 complex matrix and let $\vec{v} = c + di \in \mathbb{C}$. Suppose $\psi_1(A) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. We have $\iota(A\vec{v}) = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$ and $\psi_1(A)\iota(\vec{v}) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$. In order for our equation to be satisfied we must have $(ac - bd, ad + bc) = (x_1 c + x_2 d, x_3 c + x_4 d)$. Since we wish for this to hold independent of our choice of $\vec{v} \in \mathbb{C}$ we must have $x_1 = a$, $x_2 = -b$, $x_3 = b$, and $x_4 = a$ so that we find $\psi_1(A) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

What about a general $n \times n$ matrix? It turns out that larger matrices are constructed from this simplest case block by block. Proving this is a simple matter of demonstrating the equality of summations within matrices. Quaternion-valued matrices may similarly be expressed as real matrices. In particular, we will be interested in representing a single quaternion q in matrix form. We show this case now for future use.

Proposition 2.4. *Let $A = (a + bi + cj + dk)$ be an arbitrary 1×1 quaternion-valued matrix. Its real-valued matrix representation is given by the matrix*

$$\sigma = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Proof. To show this, we employ the same technique used above. Fix an arbitrary vector $\vec{v} = v_1 + v_2 i + v_3 j + v_4 k$ in the quaternions $\mathbb{H}$. We first define a similar identification map $\bar{\iota} : \mathbb{H} \to \mathbb{R}^{4n}$ by the formula

$$\bar{\iota}(a_1 + b_1 i + c_1 j + d_1 k, \ldots, a_n + b_n i + c_n j + d_n k) = (a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n).$$

We now wish to show that $\bar{\iota}(A\vec{v}) = \sigma\bar{\iota}(\vec{v})$. However, it merely requires an easy computational check to verify that

$$\bar{\iota}(A\vec{v}) = \begin{pmatrix} av_1 - bv_2 - cv_3 - dv_4 \\ bv_1 + av_2 - dv_3 + cv_4 \\ cv_1 + dv_2 + av_3 - bv_4 \\ dv_1 - cv_2 + bv_3 + av_4 \end{pmatrix} = \sigma\bar{\iota}(\vec{v}).$$

□

We conclude this section with a powerful theorem from analysis called the Inverse Function Theorem.

Theorem 2.1 (Inverse Function Theorem)**.** *Suppose $f : M_1 \to M_2$ is a smooth function between manifolds with $p \in M_1$. If df_p is an invertible linear map, then there is a neighborhood U of p such that $V = f(U)$ is a neighborhood of $f(p)$ in M_2, and the restriction $f|_U : U \to V$ is a diffeomorphism.*

The proof is given in § 7 of [4].

3 Spin Groups from Lie Theory

3.1 The Connection to Covering Spaces

We are now ready to apply our knowledge of matrix groups and Lie algebra to the task of calculating the double covers of special orthogonal groups. Recall that for a topological space X, a cover is a larger space that appears locally like many disjoint copies of X through the eyes of a special map called the covering map. Useful covers are in fact globally quite different from their base spaces.

Definition 3.1. Let E and B be topological spaces, and let $p : E \to B$ be a continuous surjection. Suppose that for every point $b \in B$ there is a neighborhood U_b of b such that $p^{-1}(U_b) = \coprod_i V_i$, where for each open set V_i, the restriction map $p|_{V_i} : V_i \to U_b$ is a homeomorphism. We say that p is a **covering map** and that E is a **covering space**, or simply a **cover** for short.

For a given space, its simply connected covering space is called the *universal cover*. By definition, Spin(n) is the universal double cover of $SO(n)$. Universal covers are unique, and so in order to show that a matrix group is the spin group Spin(n) we need only show that it is the universal double cover of $SO(n)$. This comes from a result in [2] which we state here.

Theorem 3.1. *Let G be a locally path-connected topological group and let $p : L \to G$ be its universal cover. Given a point in the fiber, L admits a unique topological group structure for which p is a homomorphism.*

In other words, if we can find a double cover $p : X \to SO(n)$ for which p is a homomorphism, then we will have that X is isomorphic and homeomorphic as a topological group to Spin(n). In order for us to show that a map is a cover from a Lie-theoretic point of view, we will first need to provide an equivalent definition of a cover.

Lemma 3.1. *Let X be a Hausdorff topological space. If $A = \{x_1, x_2, x_3, \ldots, x_n\}$ is a finite set of distinct points in X then there exists a neighborhood U_i of each x_i such that all U_i are disjoint.*

Proof. Let $x_1 \in A$ and write $A_1 = A - \{x_1\}$. Since A_1 is a finite set of distinct points, it is a compact subspace of X not containing x_1. Hence by Lemma 26.4 of [3] there exist disjoint open sets U_1 and V_1 containing x_1 and A_1 respectively. In the subspace topology of V_1 we may similarly find disjoint open sets U_2 and V_2 containing x_2 and $A_2 = A_1 - \{x_2\}$ respectively. Further, sets open in V_1 are open in X since V_1 is open in X. Continuing in this way, we see that the sets $U_1, U_2, \ldots, U_{n-1}$, and V_{n-1} are disjoint neighborhoods of $x_1, x_2, \ldots, x_{n-1}$, and x_n respectively.

□

Theorem 3.2. *Let X and Y be compact topological groups, and suppose $f : X \to Y$ is a surjective homomorphism and local homeomorphism with finite kernel. Then $f : X \to Y$ is a covering map.*

Proof. Let $y \in Y$. Denote the order of the kernel by n. Now $f^{-1}(y) \subseteq X$ is a discrete subset of order n, being a coset of the kernel. Since $f : X \to Y$ is a local homeomorphism there exists a neighborhood H_x for each $x \in f^{-1}(y)$ such that the restriction $f|_{H_x} : H_x \to f(H_x)$ is a homeomorphism. In addition, by Lemma 3.1 there exists a neighborhood H'_x for each $x \in f^{-1}(y)$ such that all H'_x are disjoint. Let $U_x = H_x \cap H'_x$ so that all U_x are disjoint and the restriction $f|_{U_x} : U_x \to f(U_x)$ is a homeomorphism for each $x \in f^{-1}(y)$. Since $f^{-1}(y)$ is finite, the set $W_y = \bigcap_x f(U_x)$ is open in Y and certainly contains y. Now let $V_x = (f|_{U_x})^{-1}(W_y)$. We claim that $f^{-1}(W_y) = \coprod_x V_x$ so that W_y is an evenly-covered neighborhood of y.

If $x \in V_x$ then $f|_{U_x}$ sends x into W_y and so certainly $f(x) \in W_y$, giving $\coprod_x V_x \subseteq f^{-1}(W_y)$. Now suppose there exists $a \in X - \coprod_x V_x$ such that $f(a) \in W_y$. Then $f(a) = f(b_x)$ for distinct b_x in each V_x. Since $|\ker(f)| = n$, there are n sets V_x and n distinct b_x, but because $a \in X - \coprod_x V_x$ we know that $a \neq b_x$ for any b_x. However, f is a homomorphism, so it is an n-to-one map, but a in addition to each of the n elements b_x map to the same element, which is a contradiction. Hence $f^{-1}(W_y) \subseteq \coprod_x V_x$, and so $f^{-1}(W_y) = \coprod_x V_x$.

□

3.2 Revisiting Spin(3)

In [2] we have seen that S^3 is the universal double cover of $SO(3)$, hence Spin(3) is S^3. We will now look at this in a new way by showing that the adjoint map gives rise to the double cover of $SO(3)$.

Definition 3.2. Let G be a matrix group of dimension n. The **adjoint action** of G is the map $Ad : G \to GL_n$ that sends a matrix $g \in G$ to the matrix representing the adjoint map $Ad_g : L(G) \to L(G)$.

It is easy to show that the adjoint action is a homomorphism. Let $g_1, g_2 \in G$ and let $v \in L(G)$. We have

$$Ad(g_1g_2)(v) = Ad_{g_1g_2}(v) = g_1g_2v(g_1g_2)^{-1} = g_1(g_2vg_2^{-1})g_1^{-1} = (Ad_{g_1} \circ Ad_{g_2})(v).$$

By Proposition 2.2 we have that $\mathrm{Sp}(1) \approx S^3$, hence we are led to investigate what the adjoint action does to $G = \mathrm{Sp}(1)$. The adjoint map Ad_g for an element $g \in \mathrm{Sp}(1)$ is the linear map given by conjugation of a vector in $L(\mathrm{Sp}(1)) = \mathrm{Span}\{i, j, k\}$ by a unit length quaternion. This will not alter the norm of such a vector, hence the image of Ad lies inside $O(3)$. Since Sp(1) is path connected and contains the identity $I = 1$, its image under Ad will not contain portions of the component of $O(3)$ containing matrices with determinant -1. Hence we may narrow the image under Ad down to $SO(3)$ so that we now have $Ad : \mathrm{Sp}(1) \to SO(3)$. We are now about ready for Spin(3), but first we state a minor proposition, which will prove useful.

Proposition 3.1. *Let $q \in \mathbb{H}$ with q unit length. If q commutes with every quaternion, then $q = \pm 1$.*

The proof is easily verified by a simple, though tedious, computation.

Theorem 3.3. *The adjoint action of* Sp(1) *is the double cover of* $SO(3)$ *and hence* Spin(3) *is* $\mathrm{Sp}(1) \approx S^3$.

Proof. Both Sp(1) and $SO(3)$ are compact topological groups, and we have shown that Ad is a homomorphism. By Theorem 3.2 it remains to show that Ad is a surjective local homeomorphism with kernel of order two. Suppose $g \in \ker(Ad)$, so that conjugation by g is the identity rotation in $SO(3)$. Hence by Proposition 3.1, g must be real, leaving $g = 1$ and $g = -1$. Thus the kernel has order two.

We will use the Inverse Function Theorem to show that Ad is a local diffeomorphism and hence a local homeomorphism. Because of the homogeneous nature of topological groups, it will suffice to show that Ad is a local diffeomorphism at the identity $1 \in \mathrm{Sp}(1)$. By the Inverse Function Theorem, we must show that $d(Ad)_1 : L(\mathrm{Sp}(1)) \to L(SO(3))$ is invertible. This is done by showing that $d(Ad)_1$ sends the basis $\{i, j, k\}$ of $L(\mathrm{Sp}(1))$ to a basis of $L(SO(3))$. Recall that $L(SO(3))$ is the vector space of skew-symmetric 3×3 matrices; a basis is given in Proposition 2.3.

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{Sp}(1)$ be a differentiable path such that $\gamma(0) = 1$ and $d\gamma_0 = i$. We must compute what $d(Ad)_1(d\gamma_0)$ looks like as a matrix in order to verify that it is an element of a basis for $L(SO(3))$. To do this we see how it transforms basis elements of $\mathbb{R}^3$. This is done more concisely if we identify $\mathrm{Span}\{i, j, k\}$ as $\mathbb{R}^3$ since we already know that they are isomorphic vector spaces. Upon making this identification, let $\vec{v} \in \mathbb{R}^3$. We have

$$d(Ad)_1(d\gamma_0)(\vec{v}) = \frac{d}{dt}\Big|_{t=0} Ad_{\gamma(t)}(\vec{v}) = [i, \vec{v}] = i\vec{v} - \vec{v}i = \begin{cases} 0, & \vec{v} = i \\ 2k, & \vec{v} = j \\ -2j, & \vec{v} = k\ . \end{cases}$$

By seeing what happens to the basis elements i, j, and k, we see that

$$d(Ad)_1(i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Repeating this process yields

$$d(Ad)_1(j) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d(Ad)_1(k) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 2.3, these three matrices form a basis for $L(SO(3))$. Hence Ad is a local diffeomorphism.

To show that Ad is surjective, we use arguments from point set topology. The group Sp(1) is compact, so $Ad(\text{Sp}(1))$ is also compact. Since $SO(3)$ is certainly Hausdorff, $Ad(\text{Sp}(1))$ must be closed. However, Ad is a local diffeomorphism, so its image is also open. By the connectedness of $SO(3)$ we therefore have that $Ad(\text{Sp}(1))$, being non-empty, must be the whole space.

□

Since we now know that Spin(3) is $\text{Sp}(1) \approx S^3$, we have easy access to a matrix representation. The elements of Spin(3) are all 1×1 unit length quaternion-valued matrices. Further, we may now produce equivalent real matrices out of these by applying Proposition 2.4. Thus for a quaternion $q = a + bi + cj + dk$ we have

$$q = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Hence we have that Spin(3) is the group of all such matrices for which $a^2 + b^2 + c^2 + d^2 = 1$. This is exactly what we found in [2].

3.3 Computing Spin(4)

For the case of Sp(1), the adjoint action sends a unit length quaternion q to the rotation in $\mathbb{R}^3$ that is obtained from conjugation by q. We wish to extend this idea to produce rotations in $\mathbb{R}^4$ and hence elements of $SO(4)$.

Since $\mathbb{R}^4 \cong \mathbb{H}$, we may think of the range of our desired cover as being isometries on $\mathbb{H}$. As we saw for the case of Sp(1), conjugation is an obvious choice for such an isometry, but we will clearly need something new for Spin(4). Investigating approaches very similar to conjugation leads to defining the double action map.

Definition 3.3. Let $w \in \mathbb{H}$ and let $q_1, q_2 \in \text{Sp}(1)$. The **double action map** $P : \text{Sp}(1) \times \text{Sp}(1) \to GL_4$ is defined by $P(q_1, q_2)(w) = q_1 w q_2^{-1}$, where multiplication is quaternionic.

In other words, the double action map sends a pair of unit length quaternions (q_1, q_2) to the linear map defined by left multiplication by q_1 with simultaneous right multiplication by q_2^{-1}. Because the norms of both q_1 and q_2 are one, each $P(q_1, q_2)$ is an isometry of $\mathbb{H}$. For this reason we have that the image of P is a subset of $O(4)$. However, just as in the case for $Ad : \text{Sp}(1) \to SO(3)$, since $\text{Sp}(1) \times \text{Sp}(1)$ is connected and $(1, 1) \mapsto I$ we have $P : \text{Sp}(1) \times \text{Sp}(1) \to SO(4)$.

Proposition 3.2. *The double action map is a homomorphism.*

Proof. Let $(q_1, q_2), (w_1, w_2) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and let $v \in \mathbb{H}$. We have

$$\begin{aligned} P((q_1, q_2) \cdot (w_1, w_2))(v) &= P(q_1 w_1, q_2 w_2)(v) \\ &= q_1 w_1 v (q_2 w_2)^{-1} \\ &= q_1 w_1 v w_2^{-1} q_2^{-1} \\ &= P(q_1, q_2)(w_1 v w_2^{-1}) \\ &= P(q_1, q_2) P(w_1, w_2)(v). \end{aligned}$$

□

Theorem 3.4. *The double action map is the universal double cover of $SO(4)$ and hence* $\mathrm{Spin}(4)$ *is* $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \approx S^3 \times S^3$.

Proof. Both $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and $SO(4)$ are compact topological groups, and we have shown that $P : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to SO(4)$ is a homomorphism. It remains to show that P is a surjective local homeomorphism with $|\ker(P)| = 2$.

Suppose $(q_1, q_2) \in \ker(P)$, so that for all $w \in \mathbb{H}$ we have $q_1 w q_2^{-1} = w$. Thus with $w = 1$ we have $q_1 = q_2$. By Proposition 3.1 we now have that $(q_1, q_2) = (1, 1)$ or $(q_1, q_2) = (-1, -1)$ and hence the kernel is of order two.

We will again use the homogeneous nature of topological groups to show that P is a local diffeomorphism by invoking the Inverse Function Theorem at the identity. We will show that $dP_{(1,1)}$ sends the basis $\mathcal{B} = \{(i, 0), (j, 0), (k, 0), (0, i), (0, j), (0, k)\}$ of $L(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$ to a basis of $L(SO(4))$ and hence $dP_{(1,1)} : L(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \to L(SO(4))$ is invertible.

Let $\vec{v} \in \mathbb{H}$ and let $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ be the path defined by $\gamma(t) = (e^{it}, 1)$ so that $\gamma(0) = (1, 1)$ and $d\gamma_0 = (i, 0)$. We have

$$dP_{(1,1)}(d\gamma_0)(\vec{v}) = \frac{d}{dt}\Big|_{t=0} P(e^{it}, 1)(\vec{v}) = \frac{d}{dt}\Big|_{t=0} e^{it}\vec{v} = i\vec{v} = \begin{cases} i, & \vec{v} = 1 \\ -1, & \vec{v} = i \\ k, & \vec{v} = j \\ -j, & \vec{v} = k, \end{cases}$$

which gives

$$dP_{(1,1)}(i, 0) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Repeating this process for each element in $\mathcal{B}$ we obtain the following matrices which, as the reader may verify, form a basis for $L(SO(4)) = \{A \in M_4 \mid A + A^T = 0\}$:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Lastly, the double action map P is surjective for the same reasons that $Ad : \mathrm{Sp}(1) \to SO(3)$ is. Hence P is the double cover of $SO(4)$ and Spin(4) is given by $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \approx S^3 \times S^3$.

□

3.4 Conclusion

We have now seen alternative methods for calculating and demonstrating spin groups as double covers of the special orthogonal groups. We have additionally calculated Spin(4). For an element of Spin(4), its matrix representation is given by

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A and B are of the same form as Spin(3) matrix representations.

The techniques used in this paper invoke the Lie theory of matrix groups to compute covering maps as local diffeomorphisms. It is the compactness of the matrix groups that makes this equivalence possible. We also see that the Inverse Function Theorem is a powerful tool in analyzing maps, and it was the Lie theory that allowed us to understand how the derivative of a map between matrix groups behaved. One of the benefits of this strategy is that by starting with matrix groups, one has easy access to matrix representations from the start. However, this comes with certain costs. The Lie theory used acts as a method to verify a suspected candidate for Spin(n) but offers little aid in generating the group from scratch.

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