Retracts in Category Theory

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Abstract

We first discuss retract functions in topology and give various examples in different types of spaces. We then explain many of their properties and characterize special types of known retracts. Next we develop a background in the basics of category theory in order to generalize retracts to the categorical level of abstraction. Finally, we apply our general retract definition to other common categories and explore the differences between them.

1 Introduction

A theme in modern mathematics is to study the relationships between objects instead of just focusing on the objects themselves. This allows for a greater understanding of the structure of these objects than would be possible otherwise. We will be discussing a relation called a *retract* function in the category of topological spaces. Essentially, these are functions that condense a space into one of its subspaces while leaving the subspace untouched. To picture this, imagine a piece of paper folded in half; the first half has been condensed onto the second without the second having been moved at all. These functions are more important than they might initially seem; as we will see later, they have many interesting properties. For instance, retracts respect any property preserved by surjective maps, but they also respect other invariants beyond these. In addition, retracts are intimately related to continuous extensions of functions.

Another motivation for studying retracts is that it can help elucidate the similarities and differences between different categories. After describing retracts in the category of topological spaces, we will generalize the definition of retract to categorical form. This will allow us to describe retracts in various categories such as sets, groups, and rings and understand what happens in each category. As we will see later, retracts will have very different characterizations in different categories, and thus shed light on why each category is distinct in its structure. This is where studying the relationships between different mathematical objects really shines; the language of category theory illuminates properties that would ordinarily be left in the dark by exclusively studying objects.

2 Retracts in Topology

2.1 General Properties

We begin by defining retract functions in topology.

Definition 2.1. Let X be a space and let $A \subseteq X$ be given the subspace topology. A *retraction*, or *retract*, is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Alternately stated, for the inclusion $i : A \hookrightarrow X$ where $i(a) = a$, a continuous map r is a retract if $r \circ i = 1_A$. In this case, we say that A is a *retract* of X.

As we will see later, the second definition is preferred when we begin to introduce categories because the language of categories is spoken in functions rather than points. In order to clarify the idea, a retract can be pictured in the following way:

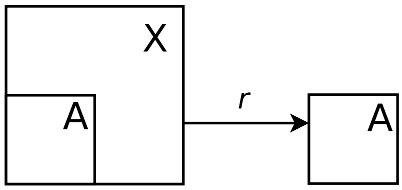


Figure 1: A retract in a general space

We will implicitly identify the image of an injective map $j : A \to B$ with A. This poses no problem since A and the image of j (as a subspace of B) will be homeomorphic. This has the effect of relaxing the definition of a retract $r : X \to A$ to require that $r \circ i = 1_A$, where $i : A \to X$ is an injective map (instead of a strict inclusion). An example of a retract follows.

Example 2.2. Let $r : \mathbb{R}^2 \to \mathbb{R}$ be such that $r(x, y) = (x, 0)$, where we have identified $\mathbb{R}$ with $\mathbb{R} \times \{0\}$. Essentially, this function projects the plane onto the real line. Note this is a retract since for any point $(x, 0)$ on the x-axis, $r(x, 0) = (x, 0)$ by default. We can picture this as follows:

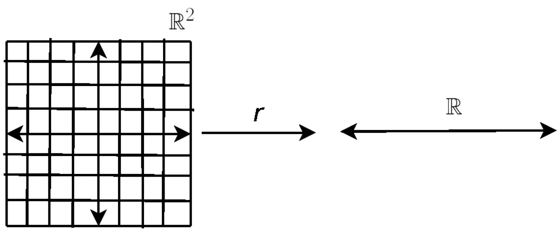


Figure 2: Retracting the plane onto the line

Retracts are not just confined to commonly encountered spaces. The following example illustrates a retract on a Möbius band.

Example 2.3. Consider the Möbius band M. Without writing the technical definition, note that we can define a continuous function $r : M \to S^1$ that "squishes" the outsides of the band onto its center circle. We can picture this as follows:

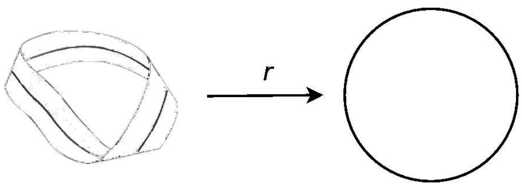


Figure 3: Retracting the Möbius band to its center circle

We can see that the center circle will not be moved in a similar fashion to the x-axis in Example 2.2. Thus we have an example of a retract.

Just as retracts can happen on exotic spaces like the Möbius band, they can even exist on a space as simple as a three-point set.

Example 2.4. Define $X = \{a, b, c\}$ as a space with open sets $\{a\}, \{b, c\}, \emptyset$, and X. Note that this defines a topology on X, so we may give $A = \{a, b\}$ the subspace topology with open sets $\{a\}, \{b\}, \emptyset$, and A. Define $r : X \to A$ by

$$\begin{aligned} r(a) &= a \\ r(b) &= b \\ r(c) &= b, \end{aligned}$$

which is continuous since $r^{-1}(\{a\}) = \{a\}$ and $r^{-1}(\{b\}) = \{b, c\}$. Thus we have a retract as r fixes a and b pointwise.

Now that we have seen a few examples of retracts in topology, we are ready to explore a few of the interesting properties they possess.

Proposition 2.5. *Retracts are always surjective.*

Proof. Note that each $a \in A$ is hit by itself since $r(a) = a$ by definition of a retract. □

An important thing to notice is that by the previous result we can conclude that retracts will automatically preserve properties such as compactness and connectedness; this is because they are already inherited by arbitrary surjective maps. Along those lines, retracts can respect other properties too.

Proposition 2.6. *Retracts respect the fixed point property.*

Proof. Let X be a space with the fixed point property, and suppose that $r : X \to A$ is a retract. We want to show that A also has the fixed point property. Let $f : A \to A$ be a continuous function on A, and consider the composition $i \circ f \circ r : X \to X$ where $i : A \to X$ is the inclusion. Since X has the fixed point property, we know there must be some $x_0 \in X$ such that

$$i(f(r(x_0))) = x_0.$$

Note if we apply r to both sides we have

$$r(i(f(r(x_0)))) = r(x_0).$$

Since $r \circ i = 1_A$ by definition of retract, it follows that

$$f(r(x_0)) = r(x_0).$$

Thus we know that f fixes $r(x_0)$, so A has the fixed point property. □

We can also consider multiple, nested retracts.

Proposition 2.7. *A retract of a retract is also a retract.*

Proof. Assume for $B \subseteq X$ we have a retract $r_1 : X \to B$ with injection $i_1 : B \to X$ such that $r_1 \circ i_1 = 1_B$. Also assume for $A \subseteq B$ we have a retract $r_2 : B \to A$ with injection $i_2 : A \to B$ such that $r_2 \circ i_2 = 1_A$. We want to show that we can retract X directly onto A. Note with $r_2 \circ r_1 : X \to A$ and $i_1 \circ i_2 : A \to X$, we can see that

$$\begin{aligned} r_2 \circ r_1 \circ i_1 \circ i_2 &= r_2 \circ (r_1 \circ i_1) \circ i_2 \\ &= r_2 \circ 1_B \circ i_2 \\ &= r_2 \circ i_2 \\ &= 1_A. \end{aligned}$$

Thus $r_2 \circ r_1$ is a retract from X to A. □

As we work with retracts, one interesting thing we will see develop over the course of this paper is that retracts are greatly intertwined with *idempotents*. Idempotents are maps $f : X \to X$ such that $f \circ f = f$.

Proposition 2.8. *A retract to a subspace induces an idempotent.*

Proof. Assume we have a retract $r : X \to A$ where $i : A \to X$ is the inclusion. We claim that $i \circ r : X \to X$ is an idempotent. If we apply $i \circ r$ twice, we have

$$\begin{aligned} (i \circ r) \circ (i \circ r) &= i \circ (r \circ i) \circ r \\ &= i \circ 1_A \circ r \\ &= i \circ r. \end{aligned}$$

Thus $(i \circ r)^2 = i \circ r$, that is, $i \circ r$ is an idempotent. □

Retracts are not only related to idempotents. As the following result will show, retracts are also related to continuous extensions of functions. Given a subspace A of X, a *continuous extension* of a map $f : A \to Y$ is a continuous function $F : X \to Y$ with $F \circ i = f$, where $i : A \to X$ is the inclusion.

Proposition 2.9. *Let $A \subseteq X$. Then A is a retract of X if and only if every continuous map $f : A \to Y$ has a continuous extension $F : X \to Y$.*

Proof. First, assume A is a retract of X with retract $r : X \to A$, and let f be a continuous map from A into Y. Since our retract $r : X \to A$ is continuous by definition, note that we can define $F = f \circ r : X \to Y$, which will also be continuous. Then F is a continuous extension of f since we have

$$\begin{aligned} F \circ i &= (f \circ r) \circ i \\ &= f \circ (r \circ i) \\ &= f \circ 1_A \\ &= f. \end{aligned}$$

Now assume that every continuous map $f : A \to Y$ has an extension $F : X \to Y$. Then note that 1_A is a continuous function, so there must be a continuous extension $r : X \to A$. This automatically means that with the inclusion $i : A \to X$, we have $r \circ i = 1_A$. Thus r is a retraction, so A is a retract of X. □

The following shows another property of retracts.

Proposition 2.10. *If X is Hausdorff, then any retract A of X is closed.*

Proof. Let $r : X \to A$ be our retract function, and pick $x \in X - A$. Then $x \neq r(x)$ because $r(x) \in A$ by definition of retraction. Since X is Hausdorff, there exist disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ such that $x \in \mathcal{U}$ and $r(x) \in \mathcal{V}$. Note $x \in r^{-1}(\mathcal{V} \cap A)$, so $x \in r^{-1}(\mathcal{V} \cap A) \cap \mathcal{U}$. Since r is continuous, we know that $r^{-1}(\mathcal{V} \cap A)$ is open in X. We want to show that x is an interior point for $X - A$ using the open set $r^{-1}(\mathcal{V} \cap A) \cap \mathcal{U}$, so suppose on the contrary there is an $a \in A$ such that $a \in r^{-1}(\mathcal{V} \cap A) \cap \mathcal{U}$. Because $a \in r^{-1}(\mathcal{V} \cap A)$, it immediately follows that $r(a) \in \mathcal{V} \cap A$. But then $r(a) = a \in \mathcal{V}$, which is a contradiction since $a \in \mathcal{U} \cap \mathcal{V}$ even though $\mathcal{U}$ and $\mathcal{V}$ were supposed to be disjoint. Thus x is an interior point of $X - A$, so it follows that $X - A$ is open. Hence A is closed. □

This result does not imply that closed retracts necessarily come from Hausdorff spaces. For example, consider another three-point space $X = \{a, b, c\}$, but this time with open sets $\{a\}, \{b\}, \{a, b\}$, $\emptyset$, and X. If we give $A = \{b, c\}$ the subspace topology with the open set $\{b\}$, we can define a retract $r : X \to A$ by $r(a) = b, r(b) = b$, and $r(c) = c$. Now note that X is not Hausdorff because the only open set that contains c is X itself, so there is no way to split b and c into disjoint open sets. In spite of this, we can see that A is closed since $X - A = \{a\}$, which is open in X.

2.2 Types of Retracts

We would first like to mention that our definition allows for homeomorphisms to be retracts. Thus we have our first type of retract.

Proposition 2.11. *Any homeomorphism is a retract. In particular, every space is a retract of itself.*

Proof. Suppose $h : X \to Y$ is a homeomorphism. Then $h^{-1} : Y \to X$ is continuous and injective with $h \circ h^{-1} = 1_Y$, so h is a retract. □

Homeomorphism retracts are certainly not very interesting. Similarly, we have another class of automatic retracts, referred to as the *trivial* retracts. These retract a space onto a single point by sending everything in the space to that point.

Proposition 2.12. *Any point $x_0 \in X$ is a retract of X.*

Proof. The constant map $r : X \to \{x_0\}$ fits the definition since it is continuous and maps x_0 to itself. □

Knowing that we have seen many different examples of non-homeomorphism and non-trivial retracts, we would like to be able to classify other known types. Note that by re-considering Example 2.2 with $r : \mathbb{R}^2 \to \mathbb{R}$, we get a large group of retracts known as the *canonical* retracts. These retracts $r : X \times Y \to X$ condense a product space onto one of its factors.

Theorem 2.13. *For any two spaces X and Y, X is a retract of $X \times Y$.*

Proof. Define $r : X \times Y \to X$ by $r(x, y) = x$. Pick a point $y_0 \in Y$ and define an injection $i : X \hookrightarrow X \times Y$ by $i(x) = (x, y_0)$ for all $x \in X$. Note for any $x \in X$ we have

$$\begin{aligned}(r \circ i)(x) &= r(i(x)) \\ &= r(x, y_0) \\ &= x\end{aligned}$$

so $r \circ i = 1_X$. □

This shows that our original retract from $\mathbb{R}^2$ into $\mathbb{R}$ was no fluke; in fact, it follows that $\mathbb{R}^a$ is always a retract of $\mathbb{R}^a \times \mathbb{R}^b$.

There is one final class of retracts that we would like to mention, ones known as *piece* retracts. These will retract a disconnected space onto one of its separated pieces.

Theorem 2.14. *Suppose $X = A \cup B$ is a separation of X. Then A (and B) is a retract of X.*

Proof. Assume X admits a separation, that is, $X = A \cup B$ for disjoint, non-empty open subsets A, B of X. Pick an element $a_0 \in A$ and define $r : X \to A$ as

$$r(x) = \begin{cases} x, & x \in A \\ a_0, & x \notin A. \end{cases}$$

We may picture this map as follows:

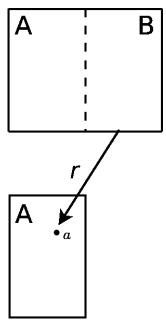


Figure 4: Retracting a space onto a piece

Note that $r(a) = a$ for all $a \in A$ by design, so we only need to show that r is continuous.

Let $\mathcal{O}_A$ be an open set in A, so $\mathcal{O}_A = A \cap \mathcal{U}$ for some open set $\mathcal{U}$ in X. Notice this implies that $\mathcal{O}_A$ is open in X since A is open in X. If $a_0 \notin \mathcal{O}_A$ then $r^{-1}(\mathcal{O}_A) = \mathcal{O}_A$, which we already know is open in X. If $a_0 \in \mathcal{O}_A$ then $r^{-1}(\mathcal{O}_A) = \mathcal{O}_A \cup B$, which is open in X since B is open in X. Thus either way $r^{-1}(\mathcal{O}_A)$ will be open in X, so r is continuous. Hence r is a retract. □

Thus we see that for any disconnected space, if we exhibit a disconnection we will create a piece retract. Extending this idea further, there is another way to phrase this result in terms of maps. Recall that a space X is disconnected if and only if there is a surjective map from X onto the zero-sphere S^0.

Corollary 2.15. *If there exists a surjective map $f : X \to S^0$, then X admits a piece retract.*

The strength of this characterization is that like our second version of the retract definition, being able to phrase the result in terms of maps will allow it to be translated into the language of categories.

Since a piece retract must be different from a homeomorphism retract by definition, a natural follow-up question to ask is whether this type of retract is structurally distinct from the trivial and canonical ones. As we will see in the following results and examples, piece retracts are indeed different from the others.

Corollary 2.16. *Assume $X = A \cup B$ for separated sets A and B. If $|X| \geq 3$, then at least one of the resulting piece retracts is non-trivial.*

Proof. Since there are at least three elements to put in the two separated sets, we know that at least one set will have more than one element. Thus if we pick that set, the resulting piece retract onto it is non-trivial. □

Now that we know piece retracts are fundamentally different from trivial retracts, we need to show that the same will hold for canonical ones too. To make this clearer, we will reconsider our three-point set in Example 2.4. In order to show that there are non-canonical retracts in this space, we will first note a few side results.

Lemma 2.17. *For a finite space X and a canonical retract A, $|A|$ divides $|X|$.*

Proof. Note that $X = A \times Y$ for some space Y since A is a canonical retract of X. Then

$$\begin{aligned} |X| &= |A \times Y| \\ &= |A| \cdot |Y|. \end{aligned}$$

□

Proposition 2.18. *Let X be a space with a prime number of elements. If $r : X \to A$ is a canonical retract then r is either a homeomorphism or a trivial retract.*

Proof. Suppose $|X| = p$, where p is prime. Since $|A|$ divides $|X|$, either $|A| = 1$ or $|A| = p$. If $|A| = 1$ then the retract is trivial, otherwise $A = X$, so the retract is a homeomorphism. □

We may return to our three-point space in Example 2.4, but given our results regarding separations, we will define a different retract on that space.

Example 2.19. Recall that $X = \{a, b, c\}$ with open sets $\{a\}, \{b, c\}, \emptyset$, and X. Note that X admits a separation into $X = \{a\} \cup \{b, c\}$. Thus we know by Corollary 2.16 that we have a non-trivial retract onto $\{b, c\}$. In addition, since $|X| = 3$, this retract must be non-canonical by Proposition 2.18. Thus we have a piece retract that is both non-trivial and non-canonical.

Now that we have shown that the previous types of retracts are structurally different, a natural question comes to mind: are there any other retracts besides the homeomorphisms and trivial, canonical, and piece ones? That is, are there retracts in connected spaces that cannot be broken down into products, even up to homeomorphism? The following commutative diagram elucidates what it means for a retract $r : X \to A$ to be canonical *up to homeomorphism*:

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} \\ Y & \xrightarrow{i'} & Y \times Z & \xrightarrow{p_1} & Y. \end{array}$$

In this diagram, h_1 and h_2 are homeomorphisms, p_1 is the projection, and $p_1 \circ i' = 1_Y$. Hence the bottom row is in fact a canonical retract. Also, as part of its structure, the diagram *commutes*, meaning that the respective arrow paths from one space to another are equal; thus for example, $h_2 \circ i = i' \circ h_1$. The existence of such a diagram says that even though the retract $r : X \to A$ may not be intrinsically canonical, it is equivalent to a canonical retract through "consistent" homeomorphisms h_1 and h_2.

Generally, it is very hard to show that such homeomorphic product spaces do not exist. However in some cases, like the previous example with the piece retract, we are able to make that assertion with a careful argument. We start with one of our original examples of a retract, the Möbius band.

Example 2.20. Recall our retract on the Möbius band, which is a continuous function $r : M \to S^1$ that "squishes" the band onto its center circle. We automatically see that this retract is not a homeomorphism and is non-trivial, and since M is connected, it cannot be a piece retract. Thus we need to show that r is not canonical, that is, $M \not\cong S^1 \times E$ for some space E. Fortunately, it is a well-known fact that the Möbius band cannot be written as a direct product of a circle and any other space. Thus r is non-canonical, so we have a previously uncharacterized type of retract.

The Möbius band is not the only example. We will now consider a retract on the real line with the standard topology.

Example 2.21. Consider $\mathbb{R}$, the real line with the standard topology. Take $A = [0, \infty) \subseteq \mathbb{R}$ and $r : \mathbb{R} \to A$ as follows:

$$r(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0. \end{cases}$$

Now consider the first type of basic open set (a, b) in A. Notice that 0 can never be included in this interval, so we have

$$r^{-1}(a, b) = (a, b),$$

which is open in $\mathbb{R}$. With the other type of basic open set, which looks like $[0, b)$, we see that

$$r^{-1}([0, b)) = (-\infty, b)$$

which is open in $\mathbb{R}$, so r is continuous. Note that for any $x_0 \in A$, $r(x_0) = x_0$ by definition, so we instantly have a retract. This is certainly neither trivial nor a homeomorphism, but can we be sure that it is not canonical, that is, $\mathbb{R} \not\cong A \times E$ for any space E?

In order to show this, we will assume by contradiction that $\mathbb{R} \cong A \times E$ for some space E. We know that $\mathbb{R} \ncong A$ since A is half-open, so we can assume that E has at least two points. A standard result of point-set topology says that the factors in a product must be path-connected if the product is path-connected. Hence as $\mathbb{R}$ is path-connected, the same must be true of E. We now note a property of $\mathbb{R}$, which is that if we remove a point x_0 from $\mathbb{R}$, then $\mathbb{R} - \{x_0\}$ is no longer connected, so it cannot be path-connected. However, we will now show that the same will not hold true for $A \times E$. That is, for any $x \in A$ and $e \in E$, we will show that $A \times E - \{(x, e)\}$ is still path-connected.

Let (a, b) and (c, d) be elements of $A \times E - \{(x, e)\}$. In order to show that there is always going to be a path between these elements, there are three cases we need to take into account.

Case 1. $a = c \neq x$ or $b = d \neq e$
Suppose that $a = c \neq x$. As $\{a\} \times E \cong E$ is path-connected we may find a path $(a, b) \to (c, d)$ in this coordinate slice. Since $a = c \neq x$ this is a valid path in $A \times E - \{(x, e)\}$. The argument in the case that $b = d \neq e$ is similar.

Case 2. $a = c = x$ or $b = d = e$
Suppose that $a = c = x$. It follows that $b \neq e$ and $d \neq e$. Since $A = [0, +\infty)$ we may pick a point $f \in A$ different from x. We may then construct a three-legged path in $A \times E$

$$(a, b) \to (f, b) \to (f, d) \to (c, d)$$

using the path-connectedness of $A \times \{b\}$, $\{f\} \times E$, and $A \times \{d\}$ respectively. It is easy to see that (x, e) is on none of these three legs, so the entire path gives a path in $A \times E - \{(x, e)\}$. The argument when $b = d = e$ is completely analogous, using the fact that $|E| \geq 2$ to pick a point $f \neq e$ in E.

Case 3. $a \neq c$ and $b \neq d$
We can again use the fact that our coordinate slices are path-connected to construct two distinct paths $(a, b) \to (c, d)$ in $A \times E$:

- $(a, b) \to (c, b) \to (c, d)$
- $(a, b) \to (a, d) \to (c, d)$.

It is not hard to check that (x, e) cannot lie on both of these paths. Hence one of these paths avoids (x, e), and so it gives a suitable path in $A \times E - \{(x, e)\}$.

Therefore $A \times E - \{(x, e)\}$ is path-connected and hence connected. This leaves us with a contradiction since $\mathbb{R} - \{x_0\}$ is disconnected, so it follows that $\mathbb{R} \ncong A \times E$. Hence our retract r is non-canonical.

Thus we have another non-canonical, non-piece retract. However, to put the problem of showing that a retract is non-canonical in perspective, what happens if we modify the previous example slightly?

Example 2.22. Consider $\mathbb{R}_l$, the real line with the lower limit topology. Take $[0, \infty) \subseteq \mathbb{R}_l$ and define a retract the same way as in the previous example. Like that example, this is certainly neither trivial nor a homeomorphism. How are we to tell if this is canonical or not? Since $\mathbb{R}_l$ is not path-connected, we cannot reuse the previous proof. We are currently unsure if this retract is canonical or not; the case-by-case nature of this process should illustrate why in general answering this question is hard.

Finally, we want to reconsider our three-point space from Example 2.4 with our original retract. Recall that we retract our set $X = \{a, b, c\}$ to the subspace $A = \{a, b\}$. Using Proposition 2.18, note that our retract r is neither trivial nor canonical. Since $\{c\}$ is not open, $X = A \cup \{c\}$ is not a separation, so we know that this retract cannot be a piece one. Thus we have a retract on a disconnected space that is not a piece retract. Therefore, we can conclude that disconnected spaces can have non-piece retracts. Like the Möbius band, this example does not fall into our previously defined classes of retracts.

Now that we have seen all of these examples of different retracts, we can confidently answer the question from before that there are non-canonical retracts. However, two nagging questions still remain. First, are all of these new unclassified retracts of the same characterization? That is, do they have more in common than just being different than our previously known types of retracts, or are they also of a fundamentally different structure from each other? And regardless of the answer to that question, more and more questions are spawned in the same vein: do there exist other types of retracts that are different from those? We currently do not have answers to these questions, so we can see why in general characterizing retracts is a hard problem in topology.

3 Category Theory

3.1 Basics

In order to generalize the notion of a retract in the context of different mathematical structures, we first need to understand some basic category theory. This will enable us to speak about retracts in a very general situation and then find an appropriate analog within each structure.

Definition 3.1. A *category* $\mathcal{C}$ consists of the following:

- A class of *objects*.
- A set $\text{Mor}_{\mathcal{C}}(X, Y)$ of *morphisms* for each pair (X, Y) of objects. Individual morphisms in this set will usually be denoted as $f : X \to Y$ instead of $f \in \text{Mor}_{\mathcal{C}}(X, Y)$.
- A *composition* defined for each triple of objects (X, Y, Z):

$$\circ : \ \text{Mor}_{\mathcal{C}}(X, Y) \times \text{Mor}_{\mathcal{C}}(Y, Z) \to \text{Mor}_{\mathcal{C}}(X, Z)$$

  where $g \circ f$ denotes the image of (f, g).

In addition, there are two special restrictions that must hold. They are as follows:

- *Associativity*: Given $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.
- *Identities*: For each object X, there is a morphism $1_X : X \to X$ such that $f \circ 1_X = f$ for all morphisms $f : X \to Y$ and $1_X \circ g = g$ for all morphisms $g : Y \to X$.

Before we get into general results regarding categories, it would be useful to go through a list of examples. Note that the name for most categories is based on the name of the objects.

Example 3.2. *Sets*
This is one of the most basic categories. The objects are sets, with the morphisms being ordinary functions between sets. Composition is ordinary function composition, which is associative and has identity maps, so this is an example of a category.

Example 3.3. *Groups*
In this example the objects are groups, with the group homomorphisms as morphisms. Then composition is defined as the composition of two group homomorphisms, which is still a homomorphism. Note that homomorphism composition is associative and that the identity function is always a homomorphism, so we have a category.

A natural question would be to ask if we could make a category out of more restrictive things like abelian groups; this turns out to be the case.

Definition 3.4. A *subcategory* of a specific category is given by a subclass of objects and subsets of morphism sets that are closed under the inherited composition with the same identities.

Note that since the morphisms are part of another category, they automatically satisfy associativity; if we make sure to include the identities and check the closure of composition, then all categorical properties will be satisfied. Therefore we are able to say that abelian groups form a subcategory of the category of groups, and also a category on their own. This is important because as we will see later, a subcategory may behave very differently than its associated supercategory.

Example 3.5. *Rings*
Given our previous definition for the category of groups, the following definition for rings shouldn't be too surprising. The objects are rings, with morphisms being the ring homomorphisms. A composition of ring homomorphisms is a ring homomorphism, so we have a category since associativity and identities hold also. Like the category of groups, we can form subcategories like commutative rings and rings with unity.

Example 3.6. *Vector spaces*
We first fix a field F. Then we have F-vector spaces, which form a category if we let the objects be the vector spaces over F and the morphisms be the linear transformations.

As we can see from the previous examples, many objects studied in algebra can be formed into categories. However, categories are not just limited to algebraic objects.

Example 3.7. *Topological spaces*
Here the objects will be topological spaces, with morphisms being continuous maps. Note that the composition of continuous maps is both continuous and associative. Also, the identity map is always continuous, so we have a category.

One common theme for all of the above categories is that the morphisms between the objects were functions, and the composition involved function composition. This certainly does not have to be the case at all, as we will see in the next examples, which are a bit more exotic.

Example 3.8. *Category of a poset*
Let P be a partially ordered set. We define the objects to be the elements of P. For $x, y \in P$, we define $\mathrm{Mor}_{\mathcal{C}}(x, y)$ to contain one morphism if $x \leq y$, and to be empty otherwise. Composition will utilize the transitivity of P, that is, for morphisms $x \leq y$ and $y \leq z$, we will get a morphism $x \leq z$. Note this is automatically associative and that the identity morphisms come from the reflexive property, so we have a category.

Example 3.9. *Category of a group*
Let G be a group. The only object will be G itself, so there can only be one morphism set, which we define to be the set of elements of G. Composition is defined as the group multiplication. Then associativity automatically holds since the multiplication came from a group, and similarly the identity morphism comes from the identity of G. Hence we have a category.

Now that we have seen many examples of categories, we are ready to describe a few properties that all categories share. As we noted with topological spaces, the notions of *injective*, *surjective* and *homeomorphic* were important for retracts, so these concepts need to be generalized to categories before we will be able to describe abstract retracts.

Definition 3.10. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$.

- The morphism f is a *monomorphism* (or *monic*) if given any two morphisms $g, h : W \to X$ with $f \circ g = f \circ h$, then $g = h$.
- The morphism f is an *epimorphism* (or *epic*) if given any two morphisms $g, h : Y \to Z$ with $g \circ f = h \circ f$, then $g = h$.

Note that it immediately follows that monic maps have the left cancellation property, and epic maps have the right cancellation property. For the categories we will be studying, "monic" will usually mean injective while "epic" usually means surjective; this is not true in the most general sense. However, there are useful results regarding injectivity and surjectivity that can be generalized to monomorphisms and epimorphisms. The following proposition abstracts a well-known result regarding functions on sets.

Proposition 3.11. *Suppose that $f : X \to Y$ and $g : Y \to X$ are morphisms in a category $\mathcal{C}$ with $g \circ f = 1_X$. Then f is monic and g is epic.*

Proof. Let $h, k : W \to X$ be morphisms such that $f \circ h = f \circ k$. Now composition with g on the left implies

$$\begin{aligned} g \circ (f \circ h) &= g \circ (f \circ k) \\ (g \circ f) \circ h &= (g \circ f) \circ k \\ 1_X \circ h &= 1_X \circ k \\ h &= k \end{aligned}$$

so f is monic. By a very similar argument, we see that g is epic. □

Now that we have that result under our belt, we can use it to make sense of the notion of categorical *isomorphism*.

Definition 3.12. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. We will call g the *inverse* of f and write $g = f^{-1}$.

The name for "isomorphism" will change depending on what category we work in; if we are in an algebraic category, then we are speaking of isomorphisms in the usual sense. In contrast, in topology we will be speaking about "homeomorphisms," and in sets we speak of "bijections."

A self-morphism $f : X \to X$ is called an *endomorphism*; it is an *automorphism* if it is an isomorphism. One interesting thing is that in Example 3.9, every morphism is an isomorphism, which is a very rare property. In addition, we mention the following result regarding inverses, which justifies the language and notation in the previous definition.

Proposition 3.13. *Inverses are unique in any category* $\mathcal{C}$.

Proof. Let $f : X \to Y$ be an isomorphism. Assume that $g, h : Y \to X$ are inverses of f. Then we know that

$$\begin{aligned} g \circ f &= 1_X & \qquad f \circ g &= 1_Y \\ h \circ f &= 1_X & \qquad f \circ h &= 1_Y. \end{aligned}$$

Note that

$$\begin{aligned} g &= 1_X \circ g \\ &= (h \circ f) \circ g \\ &= h \circ (f \circ g) \\ &= h \circ 1_Y \\ &= h. \end{aligned}$$

Hence $g = h$ and the inverse of f is unique. □

3.2 Retracts and Products

Now that we know some categorical basics, we are able to generalize the notion of retract.

Definition 3.14. A morphism $r : X \to A$ in a category $\mathcal{C}$ is a *retract* if there exists a morphism $i : A \to X$ such that $r \circ i = 1_A$.

Note that the categorical version of retract is very similar in spirit to our second definition of a retract in topological spaces. It should now be clear that this definition is preferred because it comes from the more general categorical language. However, there are a few subtleties that need to be fleshed out of categorical retracts. First, similar to what happens with topological spaces, an object can retract to other isomorphic objects. Since this is not very interesting, we will not mention it for the rest of this paper. Next, we should point out that A being a "subobject" of X is not required since categories have no notion of "element." Also, in the category of spaces our map i was the inclusion, but this no longer makes sense when talking about abstract categories; instead, by Proposition 3.11 we know that the morphism i will be monic. In addition, note that in our topological version, the map r is surjective, and by the same reasoning as with i, we know that the morphism r will be epic. Finally, make sure to recall that "epimorphism" will not always mean surjective in any given category.

Before we go any further, recall the notion of idempotent that we defined in topological spaces, which was a map $f : X \to X$ such that $f \circ f = f$. This idea can easily be extended to morphisms in categories. Interestingly enough, upon doing this we can see that just like in topological spaces, retracts in abstract categories are related to idempotents.

Proposition 3.15. *Retracts always induce an idempotent.*

Proof. Assume that $r : X \to A$ is a retract in a category with $r \circ i = 1_A$. Note that we can define the composition $i \circ r : X \to X$, which first retracts X to A and then pushes A into X. Then if we apply $i \circ r$ twice, we have

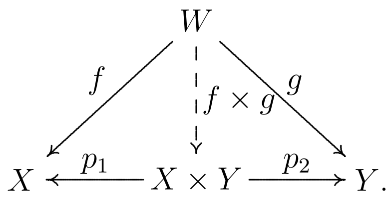
$$\begin{aligned}(i \circ r) \circ (i \circ r) &= i \circ (r \circ i) \circ r \\ &= i \circ 1_A \circ r \\ &= i \circ r.\end{aligned}$$

Thus $(i \circ r)^2 = i \circ r$, that is, $i \circ r$ is an idempotent. Hence a retract will always generate an idempotent. □

Now that we understand retracts in categories, we can make sense of the different types of retracts. First, note that the trivial retract will translate to a constant morphism to a very simple object. This will exist in some form in almost every category, so it is not very interesting, and thus will usually not be discussed. Recall that one of the other interesting topological retracts was the canonical retract, which concerned products of spaces. In order to discuss the general categorical version of this retract, we will first need to discuss the general categorical version of products.

Definition 3.16. Let X and Y be objects in a category $\mathcal{C}$. A *product* is an object $X \times Y$ with *projection morphisms* $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ that satisfy the following universal mapping property: for any object W in $\mathcal{C}$ and morphisms $f : W \to X$ and $g : W \to Y$, there is a unique morphism $f \times g : W \to X \times Y$ satisfying $f = p_1 \circ (f \times g)$ and $g = p_2 \circ (f \times g)$.

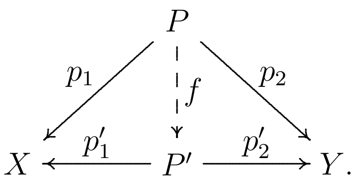
This mapping property of $X \times Y$ can be clarified by the following commutative diagram:



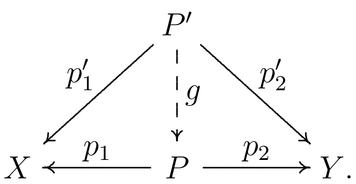
One subtlety here is that products do not exist in some categories; we will see this later when we consider the category of fields. However, we can say that if a product exists in a category, we know that it is unique up to isomorphism.

Proposition 3.17. *Products in a category are unique up to isomorphism.*

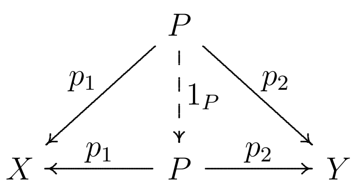
Proof. Let X and Y be two objects in $\mathcal{C}$. Suppose we have two products P, P' of X and Y (with projections p_i and p'_i). We can define a morphism $f : P \to P'$ using the product structure of P' as in the following diagram:



Reversing the roles of P and P' we may similarly define a morphism $g : P' \to P$ as in



We want to show that $f : P \to P'$ is an isomorphism with inverse g. Note that 1_P is the unique morphism making



commute. Using the diagrams above, one can check that $g \circ f$ also makes this commute, hence $g \circ f = 1_P$ by uniqueness. Similarly, one shows that $f \circ g = 1_{P'}$. □

Now that we have built up the tools to express products in general categories, we are able to express the categorical version of *canonical retract*.

Definition 3.18. A retract $r : X \to A$ in a category $\mathcal{C}$ is *canonical* if there are objects Y, Z, isomorphisms h_1, h_2, and a morphism j such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} \\ Y & \xrightarrow{j} & Y \times Z & \xrightarrow{p_1} & Y. \end{array}$$

Note. The commutativity of this diagram implies that j is monic and $p_1 \circ j = 1_Y$. Hence the bottom row is in fact a canonical retract. Moreover, X is isomorphic to $A \times Z$.

4 Retracts in Common Categories

4.1 Sets

For our first specific category beyond topological spaces, we will consider sets, one of the most basic categories. Before we get into the types of retracts, let's look at an example.

Example 4.1. Let $X = \mathbb{R}$ and $A = \mathbb{Z}$. Define $r : X \to A$ as $r(x) = \lfloor x \rfloor$, with the usual inclusion i on A. Note that for every $a \in \mathbb{Z}$, $\lfloor a \rfloor = a$, so r is a retract in the category of sets.

If the previous example seems suspiciously easy, that's because it is. In fact, in the category of topological spaces this would not be a retract since the floor function is not continuous under the standard topology on $\mathbb{R}$. As we will see in the next theorem, retracts just end up being surjective functions in the category of sets.

Theorem 4.2. *In the category of sets, a function $r : X \to A$ is a retract if and only if r is surjective.*

Proof. First we assume that r is a retract. Then r is automatically surjective since "epic" translates to "surjective" in the category of sets.

Now assume that r is surjective. Then we know that for any $a \in A$, $a = r(x)$ for some $x \in X$. Since A isn't necessarily a subset of X, define $i : A \to X$ as $i(a) = x$, where x is chosen for each a as above. Then for any $a \in A$ we have

$$\begin{aligned} (r \circ i)(a) &= r(i(a)) \\ &= r(x) \\ &= a \end{aligned}$$

so $r \circ i = 1_A$. Thus r is a retract. □

In the spirit of piece retracts that we defined in the category of topological spaces, we can note the following corollary.

Corollary 4.3. *Every set X with $|X| \geq 3$ admits a non-trivial retract.*

Proof. Let $|A| = 2$ (so that $A \neq X$) and define $r : X \to A$ such that r is surjective. By Theorem 4.2, we know that r is in fact a retract. Since $|A| = 2$, r cannot be constant, so r cannot be a trivial retract. □

The category of sets might seem odd in that there are so many non-trivial retracts, but this result is actually due to their structureless nature. As we saw previously, in every other algebraic category, morphisms have stronger conditions beyond being functions; thus with no real restrictions on the morphisms, it becomes much easier for a morphism to be a retract.

4.2 Abelian Groups

We will now consider retracts in the category of abelian groups. Applying our categorical definition, if we let A, B be abelian groups, a retract is a homomorphism $r : B \to A$ such that $r \circ i = 1_A$ for an injective map $i : A \to B$. Because of what we will see in the next few results, rather than discussing any examples of retracts involving abelian groups, we will move straight to characterizing them.

In order to discuss canonical retracts, note that in the category of abelian groups the product of two groups A, B is the direct product, written $A \times B$. This is the set of coordinate pairs (a, b) with its operation calculated component-wise. Thus a canonical retract in the category of abelian groups would be a projection homomorphism from $A \times B$ onto A or B, or more generally, a homomorphism $\phi : G \to A$ with $G \cong A \times E$ for some group E.

In order to move forward with our characterization, we note a very important result regarding idempotents and products in the category of abelian groups.

Theorem 4.4. *Let A be an abelian group with an idempotent $e : A \to A$. Then $A \cong \ker(e) \times \operatorname{im}(e)$.*

Proof. Define $\phi : A \to \ker(e) \times \operatorname{im}(e)$ by $\phi(a) = (a - e(a), e(a))$. We first note that $a - e(a) \in \ker(e)$ since for any $a \in A$ we have

$$\begin{aligned} e(a - e(a)) &= e(a) - e(e(a)) \\ &= e(a) - e(a) \\ &= 0. \end{aligned}$$

Thus we know that ϕ is well-defined.

Now for any $a, b \in A$ we have

$$\begin{aligned} \phi(a + b) &= ((a + b) - e(a + b), e(a + b)) \\ &= (a + b - e(a) - e(b), e(a) + e(b)) \\ &= ((a - e(a)) + (b - e(b)), e(a) + e(b)) \\ &= (a - e(a), e(a)) + (b - e(b), e(b)) \\ &= \phi(a) + \phi(b). \end{aligned}$$

Thus ϕ is a homomorphism.

We now want to show that ϕ is one-to-one, so we will do so by proving that ϕ has a trivial kernel. Let $x \in \ker(\phi)$. Then

$$\begin{aligned} \phi(x) &= (x - e(x), e(x)) \\ &= (0, 0). \end{aligned}$$

Therefore by the first coordinate we know that $x = e(x)$, and we also know $e(x) = 0$ by the second, so $x = 0$. Thus $\ker(\phi)$ is trivial, so ϕ is one-to-one.

Finally, we will show that ϕ is surjective, so consider $(g_1, g_2) \in \ker(e) \times \operatorname{im}(e)$. Note by definition $e(g_1) = 0$ and $g_2 = e(a)$ for some $a \in A$. Then

$$\begin{aligned} \phi(e(a) + g_1) &= ((e(a) + g_1) - e(e(a) + g_1), e(e(a) + g_1)) \\ &= (e(a) + g_1 - e(a) - e(g_1), e(a) + e(g_1)) \\ &= (e(a) + g_1 - e(a) - 0, e(a) + 0) \\ &= (g_1, e(a)) \\ &= (g_1, g_2) \end{aligned}$$

so ϕ is onto. Thus ϕ is an isomorphism, so $A \cong \ker(e) \times \operatorname{im}(e)$. □

Thus if an abelian group A admits an idempotent, A can be split apart by that idempotent. Consideration of this fact leads to a major result regarding retracts in the category of abelian groups.

Theorem 4.5. *Every retract is canonical in the category of abelian groups.*

Proof. Let A, B be abelian groups. In addition, let $r : B \to A$ be a retract, that is, $r \circ i = 1_A$ for an injective map $i : A \to B$. We can see that this automatically generates the idempotent $e = i \circ r : B \to B$ by the categorical result in Proposition 3.15. As a consequence of Theorem 4.4, ϕ provides an isomorphism $B \cong \ker(e) \times \operatorname{im}(e)$, which reduces to $B \cong \ker(e) \times \operatorname{im}(i)$ as r is surjective. Since i restricts to give an isomorphism $A \to \operatorname{im}(i)$, the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\;i\;} & B & \xrightarrow{\;r\;} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle i} \\ \operatorname{im}(i) & \longrightarrow & \ker(e) \times \operatorname{im}(i) & \longrightarrow & \operatorname{im}(i) \end{array}$$

shows that r is canonical. □

Notice that trivial retracts fall under this category since an abelian group will be isomorphic to itself crossed with the trivial subgroup. Thus this characterization is a very strong result; it completely classifies retracts in the category of abelian groups. It is also why we had no example at the beginning of this section: no matter what we chose, it would have been canonical!

In the spirit of Corollaries 2.15 and 2.16, we would like to be able to say that a surjective homomorphism onto a special group will automatically generate a non-trivial retract. Basing our intuition on the category of topological spaces, though, would lead us astray. An initial reaction would be to consider $\mathbb{Z}_2$ in the role of S^0 since they are the same as groups and as spaces. However, this is a huge problem. Note that we can find a surjective homomorphism $r : \mathbb{Z} \to \mathbb{Z}_2$ by simply taking $r(x) = x \pmod 2$, so that $\mathbb{Z}$ would then admit a non-trivial, non-isomorphism retract. But as all subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$ this cannot happen. Thus we need to find a new group, which as we can see in the following result, turns out to be $\mathbb{Z}$.

Corollary 4.6. *Let G be an abelian group. Any surjection $r : G \to \mathbb{Z}$ is a non-trivial retract.*

Proof. Assume we have such a surjection $r : G \to \mathbb{Z}$. By a special property of $\mathbb{Z}$, we can define a homomorphism $h : \mathbb{Z} \to G$ in the following way. Since r is surjective, there must be some $g_0 \in G$ such that $r(g_0) = 1$, so define $h(1) = g_0$. Thus we will automatically know how to compute $h(x)$ for any $x \in \mathbb{Z}$, that is, $h(x) = xg_0$. Note that for x, y in $\mathbb{Z}$, we have

$$\begin{aligned} h(x+y) &= (x+y)g_0 \\ &= xg_0 + yg_0 \\ &= h(x) + h(y). \end{aligned}$$

Thus h is a homomorphism. Note that for the composition $r \circ h : \mathbb{Z} \to \mathbb{Z}$, we have

$$\begin{aligned} (r \circ h)(x) &= r(h(x)) \\ &= r(xg_0) \\ &= xr(g_0) \\ &= x. \end{aligned}$$

Thus $r \circ h = 1_{\mathbb{Z}}$, so r is a retract. □

4.3 Groups

Here we will consider the category of all groups. Since abelian groups form a subcategory of groups, it should be no surprise that the definition of retract for groups will be the same as the version for abelian groups. In addition, like abelian groups, the product for groups is the direct product. However, this notion of retract will not play out as nicely as it did for abelian groups; this is mainly because we found idempotents to split in abelian groups. If we look at the proof that idempotents split an abelian group up to isomorphism, we notice that a liberal amount of commutativity was used throughout the entire proof. This does not bode well if we were to try to prove this same result in groups; in fact, as we will see in the next result, non-canonical retracts exist in the category of groups.

Theorem 4.7. *There exist non-canonical retracts in the category of groups.*

Proof. Consider the group S_3, and pick the element $(1\ 2)$, with cyclic subgroup $\langle(1\ 2)\rangle = \{(1), (1\ 2)\}$. If we let $|x|$ denote the order of $x \in S_3$, we may define a function $r : S_3 \to \langle(1\ 2)\rangle$ such that for any $x \in S_3$,

$$r(x) = \begin{cases} (1\ 2), & |x| = 2 \\ (1), & |x| = 1, 3. \end{cases}$$

In order to establish that this is a homomorphism, we will provide a case-by-case analysis for all possible pairs of elements $x, y \in S_3$.

Case 1. $x = (1)$ or $y = (1)$
Since $r(1) = (1)$, that $r(xy) = r(x)r(y)$ is immediate.

Case 2. $x = y$
Since $|x^2|$ is 1 or 3 for all $x \in S_3, r(x^2) = (1)$. Note that by definition $r(x)^2 = (1)$ no matter what x is, so again $r(xy) = r(x)r(y)$.

Case 3. $x \neq y$, both of order 2
The order of xy is always 3 in this case. Then $r(xy) = (1) = (1\ 2)^2 = r(x)r(y)$.

Case 4. $x \neq y$, both of order 3
There are only two elements of order 3 in S_3, and in fact they are inverse to each other. Hence $xy = (1)$ and so $r(xy) = (1) = r(x)r(y)$.

Case 5. $x \neq y$, one of order 2 and one of order 3
Here it is always the case that $|xy| = 2$, so $r(xy) = (1\ 2)$. However, $r(x)r(y)$ is either $(1)(1\ 2)$ or $(1\ 2)(1)$, so $r(x)r(y) = (1\ 2) = r(xy)$.

Thus $r(xy) = r(x)r(y)$ for every possible combination of elements x, y in S_3, so r is a homomorphism. As $(1\ 2)$ has order 2, r is a retract. The question is, can we have $S_3 \cong \langle (1\ 2) \rangle \times E$ for some group E? First, note that $\langle (1\ 2) \rangle \cong \mathbb{Z}_2$, so we've reduced our question to asking if $S_3 \cong \mathbb{Z}_2 \times E$. Since $|S_3| = 6$ and $|\mathbb{Z}_2| = 2$, $|E|$ must be 3. We know that every group with $|E| = 3$ is isomorphic to $\mathbb{Z}_3$, so it must be that $S_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. However, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are both abelian, which would force S_3 to be abelian, which is not the case. Thus we have found a non-canonical retract in the category of groups. □

Thus idempotents have a clear relation with retracts; since they only split abelian groups, we do not have as clear a situation for all groups. Through studying retracts we can see that adding a commutative structure changed our results greatly.

4.4 Vector Spaces

Now that we have seen that retracts in the category of abelian groups play out much more nicely than in the category of groups, one logical extension of this would be to consider what happens in vector spaces. Similarly to groups, vector spaces have direct products. In addition, vector spaces also have idempotents, which are usually called *projections*. Given this information, our first instinct would be to think that retracts will behave similarly to how they did in the category of abelian groups; this intuition turns out to be correct. Without going into much detail, we can note the following result.

Theorem 4.8. *Every retract is canonical in the category of vector spaces.*

Proof. Suppose that V is a vector space and that $e : V \to V$ is an idempotent. Because we can form an abelian group from V, e will be an idempotent from V to itself in the category of abelian groups. The map $\phi : V \to \ker(e) \times \operatorname{im}(e)$ of Theorem 4.4 will be a group isomorphism by the same proof. Note that ϕ also preserves scalar multiplication, so we see that e must split V as a vector space. Since all idempotents split, the argument of Theorem 4.5 shows that all retracts of vector spaces must be canonical. □

There is one surprising thing to take from the previous result. Because the situation with retracts is identical in abelian groups and vector spaces, the category of abelian groups is more similar to the category of vector spaces than it is to the category of groups. This is surprising because the categories of groups and abelian groups share the same morphisms. This is exactly the point of studying the relationships between mathematical objects: through studying them, we have uncovered a hidden fact that wouldn't be very obvious if we were independently studying groups and vector spaces alone.

4.5 Rings

Now that we have seen what happens in groups, we will move to the category of rings, which is a bit more restrictive. We first note an example of a retract.

Example 4.9. Consider $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients. We can define a function $r : \mathbb{Z}[x] \to \mathbb{Z}$ with $r(p(x)) = p(0)$, the constant term for the polynomial $p(x) \in \mathbb{Z}[x]$. Note that for $p(x), q(x) \in \mathbb{Z}[x]$, adding the two together will just combine the constant terms, so it automatically follows that $r(p(x) + q(x)) = r(p(x)) + r(q(x))$. Similarly, we can see that $r(p(x)q(x)) = r(p(x))r(q(x))$, so we can conclude that r is a ring homomorphism. Note that r is also surjective. Thus if we use the injection $i : \mathbb{Z} \to \mathbb{Z}[x]$, which is also a ring homomorphism, we have $r \circ i = 1_{\mathbb{Z}}$, so r is a retract.

Now that we have seen an example of a retract, we can discuss the types. Breaking with our usual protocol, we will mention trivial retracts since the situation plays out differently in special categories of rings.

Proposition 4.10. *For the subcategory of rings with unity and unity-preserving homomorphisms, there are no trivial retracts.*

Proof. Note that in order for there to be a trivial retract, there must exist a constant homomorphism, which is impossible since a unity-preserving ring homomorphism must map corresponding zero and unity elements to each other. □

Similar to the category of groups, the product in the category of rings is formed by the direct product of rings. We are interested in the prospect of having non-canonical retracts in the category of rings. Our intuition would lead us to believe that it would be similar to the situation in groups, but as we will see, this is only partly true.

Theorem 4.11. *There are non-canonical retracts in the category of rings.*

Proof. Take the retract $r : \mathbb{Z}[x] \to \mathbb{Z}$ in Example 4.9. We want to show that r is not canonical, that is, $\mathbb{Z}[x] \ncong \mathbb{Z} \times R$ for any ring R. In order to do this, we suppose on the contrary that $\mathbb{Z}[x] \cong \mathbb{Z} \times R$. Then there are two cases.

Case 1. $R = \{0\}$
If R is the trivial ring, then it must be that $\mathbb{Z}[x] \cong \mathbb{Z}$ as rings, and hence as abelian groups. However, this cannot be true since $\mathbb{Z}$ is cyclic but $\mathbb{Z}[x]$ is not.

Case 2. $R \neq \{0\}$
Then there must be some $x \neq 0$ in R. This implies that $(1, 0)$ is a zero divisor in $\mathbb{Z} \times R$ since $(0, x) \cdot (1, 0) = (0, 0)$, so $\mathbb{Z} \times R$ is not an integral domain. However, $\mathbb{Z}[x]$ is an integral domain, so they cannot be isomorphic.

Thus either way $\mathbb{Z}[x] \ncong \mathbb{Z} \times R$, so our retract is non-canonical. □

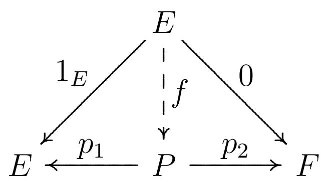
This example is extremely powerful because it exists not only in the most general category of rings, but also within the very structured subcategory of commutative rings with unity and unity-preserving homomorphisms. Thus our original intuition was correct, that groups having non-canonical retracts would likely suggest that rings would have non-canonical retracts. Unlike groups, though, note that adding a commutative structure does not change the retract situation, which is fairly surprising.

4.6 Fields

We now know that many types of rings have non-canonical retracts, so this leads us to consider the very restrictive category of fields. One immediately striking property of the category of fields is that unlike other categories we have seen so far, it does not have products. If we try to define products as direct products of fields (like rings), this will create zero divisors. Furthermore, no other construction will work.

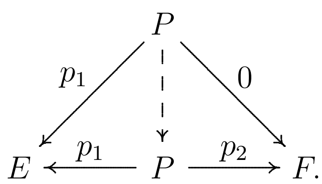
Proposition 4.12. *Products do not exist in the category of fields.*

Proof. Let E and F be two fields, and suppose that P is a product of E and F in the category of fields. Consider the diagram



where $f = 1_E \times 0$. If $p_1 = 0$ then $1_E = 0$, a contradiction. Hence $p_1 \neq 0$ and so it must be monic. Using what we know about sets alone, the equality $p_1 \circ f = 1_E$ shows that p_1 must be surjective. Hence p_1 is a field isomorphism between P and E. The same argument applies to F to show that p_2 is an isomorphism between P and F.

Now apply the universal mapping property of P to the diagram



Since p_2 is an isomorphism, the zero map is the only homomorphism making the right-hand triangle commute, so the dotted arrow must be the zero map. However, this forces $p_1 = 0$ in the left-hand triangle, a contradiction. Hence the product of E and F does not exist. □

Therefore the lack of products makes canonical retracts an impossibility. Also, by similar reasoning to Proposition 4.10, the category of fields will not have trivial retracts. There is still the possibility of other types of retracts, but as the following theorem shows, this will not be the case.

Theorem 4.13. *Every retract is an isomorphism in the category of fields.*

Proof. Since a retract must be a non-zero homomorphism, note that it must be injective since non-zero field homomorphisms are injective. However, since retracts are surjective by definition, our retract would just be an isomorphism. □

As we can see, the very defined structure of a field is too restrictive for retracts. Thus by studying retracts we can see that the categories of fields and rings (even commutative rings with unity) have very different structures. In this sense, fields are almost the exact opposite of sets; the flimsy nature of sets allowed for a giant class of retracts, while the rigidity of fields does not allow for any interesting retracts.

4.7 Categories From Posets and Groups

Although we have previously been investigating retracts in algebraic categories, this certainly does not imply that they are the only categories in which we can investigate retracts. To make this point more clear, in this short section we will consider retracts in the category of a poset and the category of a group. Both of these categories have very different structures than the algebraic categories we previously encountered.

Recall that a morphism exists between two elements x and y in the category of a poset P if $x \leq y$. By reconsidering the definition of a retract, we can easily see that the only retracts in this category are the identities.

Proposition 4.14. *Every retract is an identity in the category of a poset.*

Proof. Assume we had a retract $r : x \to a$ for two elements a, x in a poset P. Since r is a morphism, this means that $x \leq a$. Since we need a morphism $i : a \to x$ as part of the definition of retract, this means that $a \leq x$. Then $x = a$, so r is the identity. □

Now consider the category of a group. Recall that the only object is the group itself, and the morphisms are the elements of that group. Given that there is only one object and that every morphism is an isomorphism, it is far from surprising that the consideration of retracts does not go very far.

Proposition 4.15. *Every retract is an isomorphism in the category of a group.*

Though it is beyond the scope of this paper, we would like to note that the category of a poset and the category of a group are *EI-categories*. An EI-category is one in which every endomorphism is an isomorphism. It turns out that in an EI-category the isomorphisms are the only retracts. However, the converse does not hold: the category of fields is not an EI-category, yet the only retracts are the isomorphisms. Even though the results for categories formed from posets and groups are not very substantial on their own, they still provide valuable insight on how the structure of a category can drastically change the behavior of relations like retracts.

5 Conclusion

Now we have seen how retracts work in many different categories; in some categories almost anything can be a retract, where in others they rarely exist. Also, beyond studying their general properties, we have demonstrated that there are large classes of known retracts that are shared between categories. We have seen that the notion of retract is not an isolated construction, but instead is considerably related to extensions, idempotents, and products. Through studying retracts, it has hopefully been clear that there are many structural distinctions between objects like sets, groups, and topological spaces. There are many subtleties to a mathematical structure, but it takes a varied selection of tools to uncover them.

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