Finite Co-H-spaces are Contractible: A Dual to a Theorem of Stong

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Abstract

After providing a summary of known results in the homotopy theory of finite spaces, we prove that finite co-H-spaces are necessarily contractible. This result is dual—though the proof is not—to a 1966 theorem of Stong stating that finite H-spaces are similarly homotopically trivial. Together, these two results explain why infinite spaces feature so prominently in classical algebraic topology.

Introduction Nearly everyone's first exposure to algebraic topology is the construction of the fundamental group $\pi_1(Y)$ of a based space Y. This is most efficiently described as the group of homotopy classes of based maps[1] ("loops") $S^1 \to Y$ under a straightforward loop-concatenation operation. Here, S^1 denotes the 1-dimensional sphere or unit circle, and using standard notation we may write $\pi_1(Y) = [S^1, Y]$. One next learns that there is nothing special about the 1-dimensional sphere in this regard: using the same constructions we may just as easily define $\pi_n(Y)$, the *nth homotopy group of* Y, as $\pi_n(Y) = [S^n, Y]$, again with an obvious concatenation operation.

While the unit circle S^1 may not be special among the spheres in this context, the spheres themselves do enjoy some special structure. Given a fixed space X, the assignment $F_X(Y) = [X, Y]$ defines a functor F_X from the category $\mathbf{Top}_*$ of based spaces to the category $\mathbf{Sets}_*$ of based sets. To "do algebraic topology," one would prefer that F_X take values in the category **Gps** of groups. It is well-known that F_X defines a functor $F_X : \mathbf{Top}_* \to \mathbf{Gps}$ if and only if X is a *co-H-group* (definition to come). Of course, it turns out that each sphere S^n has the structure of a co-H-group, and the resulting group-valued functor F_{S^n} is just π_n. All of these questions may be dualized—fix the codomain and let the domain vary—leading to the notion of *H-group*.

In the past decade or so there has been a renewed interest in the homotopy theory of finite topological spaces; see our list of references for a quick survey. That finite spaces are homotopically interesting is a theorem of McCord from 1966: to each finite simplicial complex there is a finite space weakly homotopy equivalent to it [8]. Hence finite spaces have the same homotopy and singular homology groups as the finite simplicial complexes.

[1] For simplicity of exposition, let us agree that all spaces, maps, and homotopies in this note are implicitly based from now on.

In particular, there are *finite* spaces weakly homotopy equivalent to the n-spheres. This surprising fact has been exploited with much success to give *finite* models of classical homotopy constructions in terms of relatively simple objects, usually finite partially-ordered sets. For example, in [6] and [7] the authors provide finite poset models of the famous Hopf map $S^3 \to S^2$ and the non-trivial element of $\pi_5(S^3)$, respectively. Moreover, their techniques involve making use of finite poset analogues of the multiplicative structure on S^1, yet another feature of this space admitting a finite model.

All of this raises a tantalizing series of questions. For one, is there a finite space X with the property that F_X closely approximates π_n? After all, there is a finite space X weakly homotopy equivalent to S^n, so perhaps $F_X(Y) = [X, Y]$ approximates a fair portion of $\pi_n(Y) = [S^n, Y]$ for arbitrary Y. If so, we have gained a potentially powerful computational tool: to understand $\pi_n(Y)$, we first pass to its approximation $F_X(Y)$, then make use of the finiteness of X to control our calculations and arguments. This would be even simpler if Y itself were finite, for $F_X(Y)$ is then automatically finite. Even more generally, we can ask if there are finite spaces X for which F_X provides a group-valued functor, whether or not this may relate to π_n.

The main result of this article is that all of these questions have, unfortunately, negative answers. Thus we cannot "do algebraic topology" with F_X when X is finite. Precisely, we prove that a finite space X admits the structure of a co-H-space if and only if it is contractible. In particular, there is no hope for a co-H-group structure, so that F_X will never be group-valued (in a non-trivial way) for finite spaces X. Said differently, we prove that the only time F_X is group-valued is when $F_X(Y)$ is the trivial group for all spaces Y. This result is dual to a theorem of Stong on H-spaces (Theorem 5 of [10]), but neither our proof nor his dualizes to give the other.

H-spaces and Stong's theorem One could argue that the homotopy theory of finite spaces effectively began in 1966 with the papers by McCord ([8]) and Stong ([10]). Curiously enough, these papers were independent, each author referring to the work of the other as "to appear." In [10], Stong provides the foundational results for the analysis of the homotopy theory of finite spaces in terms of posets. As an application of this machinery, he proves the following theorem, the dual of which is the subject of this article:

Theorem (Theorem 5 of [10]). *Let X be a finite connected space. Then X admits the structure of an H-space if and only if it is contractible.*

This result is a bit surprising, and the proof is rather long and difficult. As a reminder, an *H-space* is the topologist's version of a set equipped with a binary operation having a two-sided identity. Roughly, one takes what all this would mean to an algebraist, phrases everything in terms of commutative diagrams, but then requires the diagrams to commute only up to homotopy. The basepoint of the H-space essentially acts as the identity element, so this is made part of the definition. Precisely, an H-space is a based topological space X together with a based map

$\mu : X \times X \to X$ (the "multiplication") such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{1 \times e} & X \times X & \xleftarrow{e \times 1} & X \\ & \searrow^{1_X} & \downarrow \mu & {}^{1_X}\swarrow & \\ & & X & & \end{array}$$

commutes up to homotopy. Here we are writing $e : X \to X$ for the constant map to the basepoint of X, and terms like $1 \times e$ represent canonical coordinate maps. An *H-group* is simply an H-space with additional homotopy-commutative diagrams encoding associativity and existence of inverses. The standard example of an H-group is the space ΩX of loops in a space X under ordinary loop-concatenation.

Stong's result says something very strong about finite spaces and the representable contravariant functors defined via homotopy classes of maps. If we fix the space Y and allow X to vary, the assignment $F^Y(X) = [X, Y]$ defines a contravariant functor $F^Y : \mathbf{Top}_* \to \mathbf{Sets}_*$. As before, we would prefer that F^Y take values in the category of groups, and it is well-known that F^Y defines a functor $\mathbf{Top}_* \to \mathbf{Gps}$ if and only if Y is an H-group. For finite spaces Y, Stong's theorem implies that this occurs if and only if Y is contractible, so that $F^Y(X)$ is the trivial group for all X. Our goal is to show that the dual results also hold.

Co-H-spaces Let us denote by $X \vee Y$ the *wedge sum* of the based spaces X and Y. This is usually defined as the disjoint union of X and Y with the single identification $x_0 \sim y_0$ of their respective basepoints. Equivalently, we may regard $X \vee Y$ as the subspace consisting of the "axes" in $X \times Y$, so that $X \vee Y$ consists of the ordered pairs (x, y) where at least one of the coordinate entries is the basepoint. This latter description will be more convenient for us. The essential property of wedge sums is the following easily verified fact: given maps $f : X \to Z$ and $g : Y \to Z$, the map $f \vee g : X \vee Y \to Z$ defined piecewise by

$$(f \vee g)(x, y) = \begin{cases} f(x), & x \in X \text{ and } y = y_0 \\ g(y), & x = x_0 \text{ and } y \in Y \end{cases}$$

is continuous. In fact, it is the unique such map behaving this way on the respective "pieces" X and Y.

Formally, the definition of *co-H-space* is the categorical dual to that of *H-space*, with wedge sums replacing the products and all arrows reversed. This gives us the following definition.

Definition 1. A *co-H-space* is a based space X together with a map $\eta : X \to X \vee X$ (called the *comultiplication*) making the diagram

$$\begin{array}{ccccc} & & X & & \\ & {}^{1_X}\swarrow & \downarrow \eta & \searrow^{1_X} & \\ X & \xleftarrow{1 \vee e} & X \vee X & \xrightarrow{e \vee 1} & X \end{array}$$

commute up to homotopy.

Note. The map $1 \vee e : X \vee X \to X$ acts as $(1 \vee e)(x, x_0) = x$ and $(1 \vee e)(x_0, x) = x_0$. Of course, $e \vee 1$ acts similarly.

By similarly dualizing the diagrams for associativity and inverses, we obtain the diagrams defining *co-H-groups*. (This hardly matters for our purposes, as we assert that finite spaces do not even admit interesting co-H-space structures.)

The standard example of a co-H-group is the suspension ΣX of a based space X. The suspension of X is defined by first forming the cylinder $X \times [0, 1]$ and then identifying the top and bottom copies of X along with the baseline $\{x_0\} \times [0, 1]$, all to a single point. The comultiplication $\eta : \Sigma X \to \Sigma X \vee \Sigma X$ is the map pinching the "equator" of ΣX to a point, yielding two tangent copies of itself. The n-spheres ($n \geq 1$) are suspensions (in fact, $\Sigma S^n = S^{n+1}$), and this is the reason that $\pi_n(Y) = [S^n, Y]$ forms a group. All of this is most easily seen for the case of the unit circle S^1, whereupon pinching the north and south poles we obtain the figure eight $S^1 \vee S^1$.

Homotopy theory of finite spaces Given a finite space X and a point $a \in X$, we write M_a for the intersection of all open sets in X containing a. Since X is finite, M_a is necessarily open, and we call it the *minimal open set* about a (and it is). We may define a relation $\leq$ on X by declaring $a \leq b$ if and only if $M_a \subseteq M_b$. This relation is plainly reflexive and transitive, but it is anti-symmetric if and only if X satisfies the T_0 separation axiom. Hence $(X, \leq)$ is a poset if and only if X is T_0, and for this reason most authors restrict their attention to finite T_0 spaces. For the purposes of homotopy theory this is no real restriction at all, thanks to the following result:

Proposition 2 (Theorem 4 of [8])**.** *There is a natural construction that assigns to every finite space X a quotient space $\widehat{X}$ such that:*

(i) the space $\widehat{X}$ is T_0,

(ii) the quotient map $q_X : X \to \widehat{X}$ is a homotopy equivalence, and

(iii) given a map $f : X \to Y$ of finite spaces, there is a unique map $\widehat{f} : \widehat{X} \to \widehat{Y}$ such that $q_Y \circ f = \widehat{f} \circ q_X$.

Thought about the right way, the strategy for proving this result is clear. The obstruction to X being T_0 is that $\leq$ may not be anti-symmetric, so that we may have $M_a = M_b$ yet $a \neq b$. So why not force such points to be equal? If we define a relation $\sim$ on X by $a \sim b$ if and only if $M_a = M_b$, the resulting quotient space defines $\widehat{X}$.

Suppose now that X is a finite T_0 space. A point $x_1 \in X$ is said to be *upbeat* if there exists a point $x_2 \in X$ with $x_1 < x_2$ such that whenever $x_1 < y$ we also have $x_2 \leq y$. Similarly, x_1 is called *downbeat* if there exists a point x_0 with $x_0 < x_1$ such that whenever $y < x_1$ we also have $y \leq x_0$.

Definition 3. A finite T_0 space is said to be *minimal* if it has no upbeat or downbeat points.

In [10], Stong proves that minimal spaces suffice to understand the homotopy types of finite T_0 spaces. Precisely, he proves that if $x \in X$ is either upbeat or downbeat, the subspace $X - \{x\}$ is a strong deformation retract of X. In particular, the two spaces are homotopy equivalent. By successively removing all upbeat and downbeat points, we eventually arrive at a *minimal* space X_0 that is a strong deformation retract of X.

Definition 4. Let X be a finite T_0 space. A minimal subspace that is a strong deformation retract of X is called a *core* of X.

The thrust of Stong's work in [10] is that every finite T_0 space X has a core X_0, and that this core is reasonably computable. A core of such a space X need not be unique, but any two cores of X must be homeomorphic. While it is nice to know that X_0 is a strong deformation retract of X, for our purposes we only need to recognize that X and X_0 are homotopy equivalent.

Our theorem that finite co-H-spaces are contractible (homotopy equivalent to a point) will follow from the next unusually strong result.

Proposition 5 (Theorem 3 of [10])**.** *Let X be a finite, minimal T_0 space. If $f : X \to X$ is homotopic to the identity map, then f is equal to the identity map.*

The proof of Proposition 5 makes heavy use of Stong's poset machinery, which applies nicely to finite function spaces. In particular, it applies to the space of maps from X to itself, and the path component of this mapping space containing the identity 1_X is the ordinary homotopy class of the identity map. It is this path component that is shown to be a singleton. The spirit of the argument is this: the identity map on X fixes all points, and the minimality gives no room to move to deform this, so no other maps can be homotopic to the identity.

Our main result We establish our claim on the contractibility of finite co-H-spaces by making several simplifying assumptions and then successively removing them. First, we need a standard lemma.

Lemma 6. *Suppose that X is a co-H-space and that Y is homotopy equivalent to X. Then Y also admits the structure of a co-H-space.*

Proof. By assumption, there are maps $F : X \to Y$ and $G : Y \to X$ such that $G \circ F \simeq 1_X$ and $F \circ G \simeq 1_Y$. The map F provides a new map $X \to Y \vee Y$ by the action $x \mapsto (F(x), y_0)$, as well as a mirror-image map defined by $x \mapsto (y_0, F(x))$. We may wedge these together to get a new map $K : X \vee X \to Y \vee Y$. One now checks that the composition

$$Y \xrightarrow{G} X \xrightarrow{\eta} X \vee X \xrightarrow{K} Y \vee Y$$

defines a co-H-space structure on Y. □

Proposition 7. *Suppose that X is a finite, connected, and minimal T_0 space. Then X admits the structure of a co-H-space if and only if it consists of a single point.*

Proof. The forward implication ($\Rightarrow$) is the only one requiring proof. Taking the assumptions, we will prove that every point of X is equal to the basepoint x_0. By the definition of co-H-space, $(1 \vee e) \circ \eta$ must be homotopic to the identity 1_X. However, X is minimal, so Proposition 5 shows that $(1 \vee e) \circ \eta$ is in fact equal to the identity. Likewise, $(e \vee 1) \circ \eta$ is the identity.

Take an arbitrary point $x \in X$. The element $\eta(x) \in X \vee X$ takes one of two possible forms: it is either of the form (x_0, x') or (x', x_0) for some $x' \in X$. In the first case, we have

$$((1 \vee e) \circ \eta)(x) = (1 \vee e)(x_0, x') = x_0.$$

Since $(1 \vee e) \circ \eta = 1_X$, we see that $x = x_0$. Similarly, in the second case we use that $(e \vee 1) \circ \eta = 1_X$ to deduce that $x = x_0$. In any event, we see that X consists of only one point. $\square$

At last, we have the following dual to Stong's theorem on finite H-spaces.

Theorem 8. *Suppose that X is a finite connected space. Then X admits the structure of a co-H-space if and only if it is contractible.*

Proof. As before, only the forward direction requires proof. Suppose that X admits the structure of a co-H-space. By Proposition 2 we may assume that X is already T_0. Let X_0 be a core of X. Since X_0 is homotopy equivalent to X, Lemma 6 shows that X_0 has the structure of a co-H-space. However, Proposition 7 applies to X_0, so that X_0 must be a one-point space. Thus we have that X is homotopy equivalent to a one-point space, so X is contractible. $\square$

Conclusion As short as our proof of Theorem 8 is, it unfortunately does not dualize to give a more compact proof of Stong's result on H-spaces. The length and complexity of Stong's argument is a necessity due to the nature of product spaces, whereas we get lucky with the "near-disjointness" inherent to wedge sums. Taken together, these two results explain why classical homotopy theory is represented by infinite spaces: finite spaces cannot support the structure necessary for doing algebraic topology in any non-trivial way.

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