The Borsuk-Ulam Theorem & Applications to Sphere Problems

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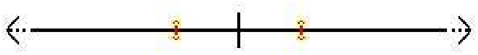
Outline

Definitions

Definition

The *n-sphere* S^n is the set of points in $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ located at a distance 1 from the origin.

For example, S^0 is two points:



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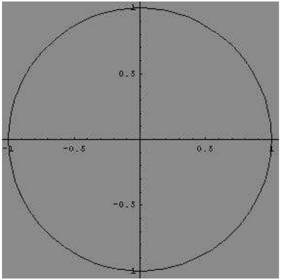
For example, S^1 is a circle:



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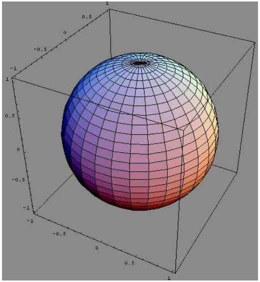
For example, S^2 is an ordinary sphere:



Definitions

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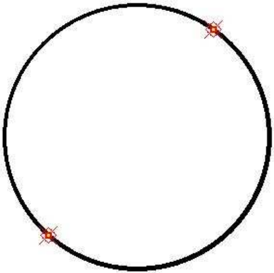
The *antipode* of a point x on a sphere is the point diametrically opposite x.

For example, a point x and its antipode $-x$ in S^1:



Our Main Problem

Rules for Coloring Spheres

1. *every point must be colored*
2. *a point and its antipode must always be different colors*

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Problem (The Coloring Problem)

We wish to find the coloring number, that is the minimum number of colors required to color a sphere.

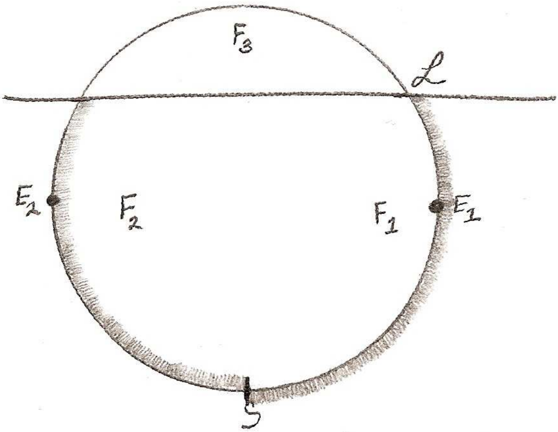
How to Solve?

How do we prove that the coloring number for a sphere is k?

- Construct a coloring that uses k colors
- Prove there is no coloring which uses only $k-1$ colors

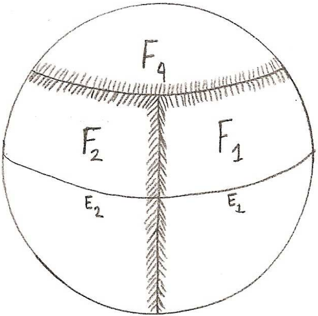
Coloring the 1-sphere

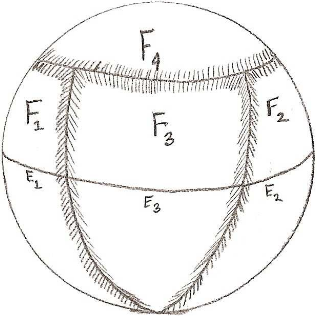
We show that the coloring number for the 1-sphere is three.



Coloring the 2-sphere

We can use the coloring of the 1-sphere to produce a coloring of the 2-sphere.





The Borsuk-Ulam Theorem

Why is there no 3-coloring of the 2-sphere?

Theorem (Borsuk-Ulam for S^2)

Given a continuous function $f : S^2 \to \mathbb{R}^2$, there is a point x of S^2 such that $f(x) = f(-x)$.

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Analogy (Theorem of Meteorology)

At any moment in time there exist antipodal points on the surface of the earth that have exactly the same pressure and temperature.

Using the Borsuk-Ulam Theorem

We cannot color the 2-sphere with three colors. Lets say we could, with Red, Green, and Blue. Then we could construct a function $f : S^2 \to \mathbb{R}^2$ by

$$\begin{aligned} f(x) = & (\textit{minimum distance between } x \textit{ and Red}, \\ & \textit{minimum distance between } x \textit{ and Green}). \end{aligned}$$

By the Borsuk-Ulam Theorem we know that $f(x) = f(-x)$ for some x. But this is impossible. Why?

Using the Borsuk-Ulam Theorem

Recall that

$$\begin{aligned} f(x) &= (min.\ dist.\ between\ x\ and\ Red, \\ &\quad min.\ dist.\ between\ x\ and\ Green) \end{aligned}$$

is to satisfy $f(x) = f(-x)$ for some x.

Consider the following three cases:
Case 1: x is Red
Case 2: x is Green
Case 3: x is Blue

A Pattern Emerges?

Lower-dimensional cases and a conjecture:

$$\begin{aligned}
\text{Coloring number for } S^0 &= 2 \\
\text{Coloring number for } S^1 &= 3 \\
\text{Coloring number for } S^2 &= 4 \\
&\vdots \\
\text{Coloring number for } S^n &= n+2
\end{aligned}$$

Main Result

In general by constructing colorings and by using the Borsuk-Ulam Theorem for higher dimensions we find that...

Theorem (Solution to the Coloring Problem)

The coloring number for the n-sphere is $n + 2$*.*

Proving the Borsuk-Ulam Theorem

Methods used to prove the Borsuk-Ulam Theorem:

- 0-sphere - Relatively Easy
- 1-sphere - Point-set Topology
- 2-sphere - Homotopy Theory
- n-sphere - Homology Theory (for $n \geq 3$)

Other Applications

We were able to use the Borsuk-Ulam Theorem to solve the following problems:

Problem (The Common Zero Problem)

If $f, g : S^2 \to \mathbb{R}^2$ preserve antipodes must they share a common zero? That is, must $f(x) = 0 = g(x)$ for some x?

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Is it possible to bisect n regions on S^n by a single great circle?

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Is it possible to bisect n regions on S^n by a single great circle?

Yes to both!

References

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